Introduction to Probability Exercises

Pages	Problems
358 – 361	1, 6, 7a, 8, 17 – 24
368 – 371	1 – 7, 23, 24, 25

Look back to exercise 1 on page 368. In that one, you found that the probability of rolling a 6 on a twelve sided die was $\frac{1}{12}$ (or, about 8%). Let's make sure that we remember what that ***means***. Which of the following two statements is true, and which is false?

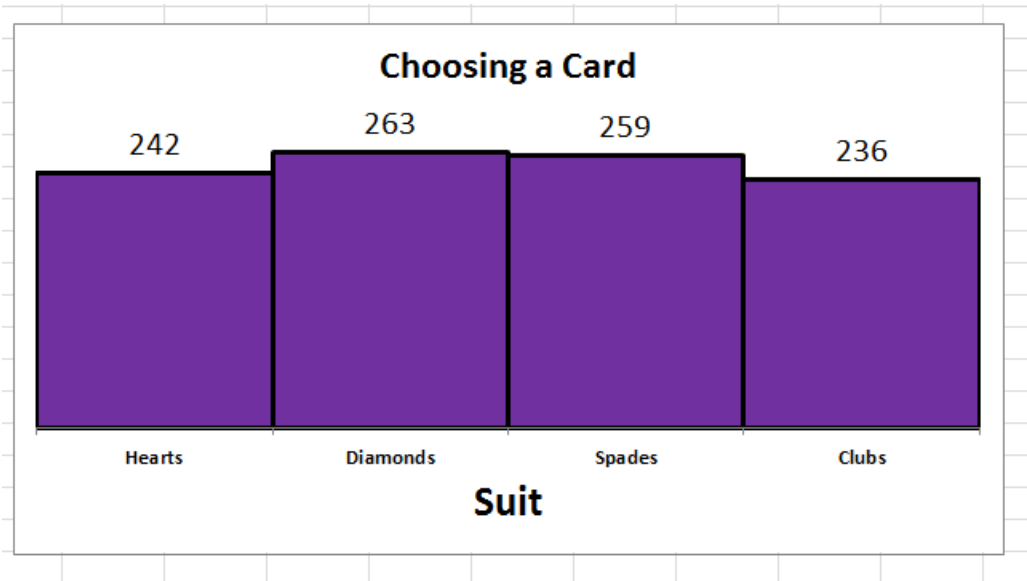
E1. If you roll that 12 – sided die 12 times, you will get 6 exactly one time.

E2. Over time, you will get 6 on one – twelfth of the rolls you roll.

E3. Refer back to page 371, exercise 25. Calculate the percent error for each of the experimental probabilities found in exercise 25.

E4. Sum the errors from **E3**. What do you notice?

I created a simulation of exercise 24, page 371, using Excel (feel free to stop by and see it!). Here are my results:



E5. How many "cards" did I "draw"?

E6. What is the percent error for each experimental probability?

E7. Give one way the errors are similar to the ones in E3.

E8. Give one way they're different.

E9. A throwback…do the "Checkpoint" at the bottom of page 388 in the Larson text (this link is better than the one they give: http://math.andyou.com/tools/montyhallsimulator/montysim.htm). What do you say? Do the results you get support the solution above the Checkpoint (and our observations from the first day)?

Answers.

E1. False. In fact, the chance of this happening is more like 40%! If you can explain why (or design a spreadsheet that demonstrates this), we'll get you some extra credit points.

E2. True! The exact definition of probability!

E3. Percent Error = $\dfrac{\textbf{observed value - "true" value}}{\textbf{"true" value}}$**.** So, I'll do the percent error in the "hearts" probability:

Percent Error = $\dfrac{\frac{17}{50} - \frac{12.5}{50}}{\frac{12.5}{50}} \approx 36\%$. That is, our observations showed "36% too many" hearts over 50 trials.

<u>Note</u>: error will be positive when you see "too many" of something and negative when you see "too few".

E4. Coincidence? Why or why not?

E5. Check out the numbers at the top of each bar!

E6. I think you can get this! Also, using Excel is a very easy way to do it. Stop into the office and I'll show you!

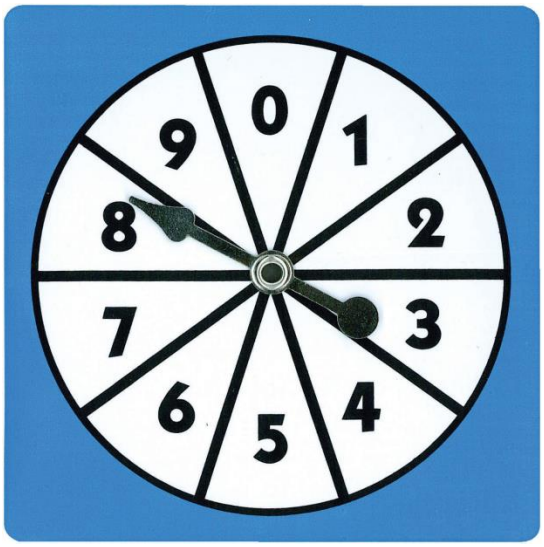
E7. <u>Hint</u>: look at E4.

E8. <u>Hint</u>: Consider the definition of probability!

E9. You got this!

Quiz 1.

Let's consider the experiment of spinning the spinner shown one time (assume that it doesn't matter toward which number the spinner arrow is pointing when you begin). Find



1. (**2 points**) the chance the spinner arrow lands on 1.

2. (**2 points**) the chance the spinner arrow lands on an even number.

OK, here's another:

The chance the spinner arrow lands on a prime[1] number is 40%.

3. (**2 points**) What does that ***mean***? One of the two statements below is more correct. Which?

   a. **Every ten times you spin the spinner, the arrow will land on a prime 4 times.**
   b. **Over many, many spins of the spinner, the arrow will land on primes about 40% of the time, on average.**

Lin case you're stuck, let's simulate! There's a spreadsheet that accompanies this quiz; go ahead and open it (but make sure you don't have any ***other*** spreadsheets open, as that goofs it up!).

Now, what you're going to do is press the F9 button to "spin" the "spinner" (it's more of a "pointer" in Excel. 😊). Each time you press it, the "spinner" "spins", and the sheet keeps track of "where the spinner landed". If you hold it down, it'll repeat this experiment many, many times. Go ahead and hold it down until you've "spun" at least 1000 times (you can keep track in cell F12).

4. (**2 points**) How many times did your spinner land on a prime number?

5. (**2 points**) (**w**) Therefore, based only on your data, what's the probability of landing on a prime number? Show me the division you used to get your percentage!

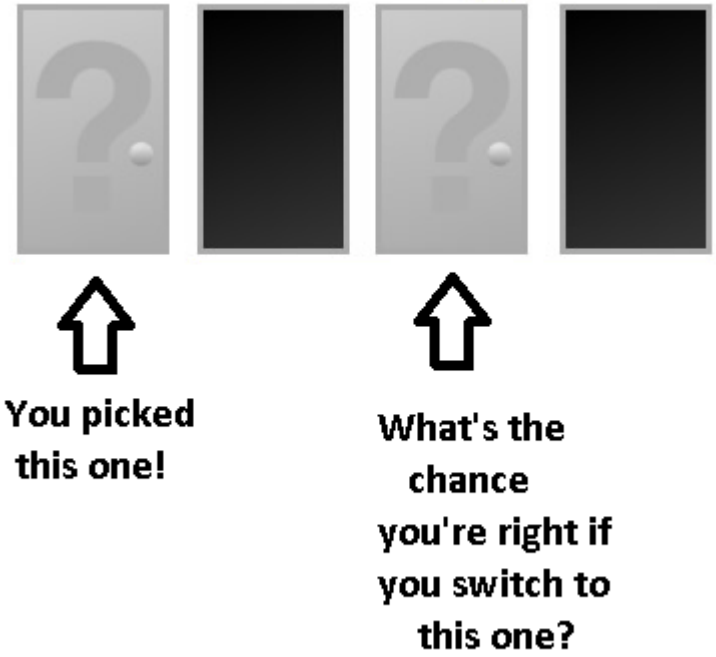
[1] By definition, a prime number is a whole number , bigger than 1, with only two factors: 1 and itself. So, from 0 to 9, the only primes are 2,3, 5 and 7.

Quiz 2.

So, in class, we discovered that, on the show Let's Make A Deal, it makes more sense to switch than stay (you'll win with probability 2/3 if you switch). Let's change the rules of the game a smidge for a couple of alternate scenarios:

Suppose there are 4 doors, instead of 3:

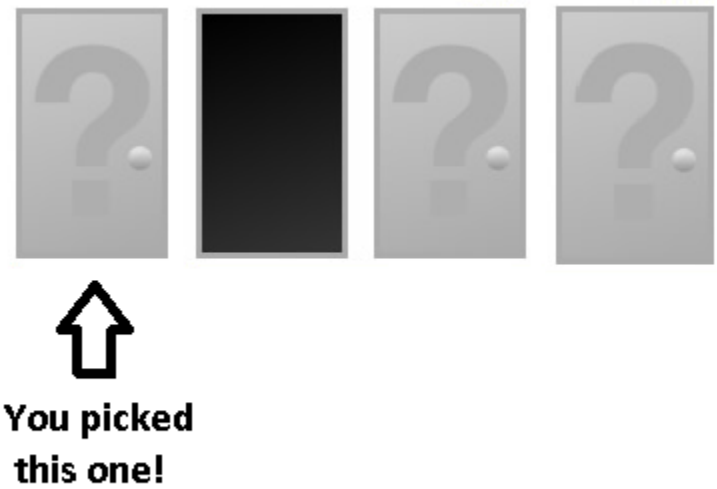
And suppose after you choose your door, Monty opens TWO empty doors, and offers you to switch:



1. (**3 points**) Clearly, you should – but what's the ***probability*** of winning if you switch? Explain your answer! Hint: ask yourself, "What's the chance you selected the correct door at the start of the game?" If you stay with that door, then the chance that you win is exactly that number (since, ***that*** percentage of the time, the prize is behind the door you selected). Once you know that percentage, you can figure out the other percentage by subtracting that percentage from 100%.

2. (**2 points**) Suppose there are 10 doors! You choose a door, and Monty opens EIGHT empty doors…what's the probability of winning if you switch? Again, explain! Hint: same logic as the last question, but (hopefully) made even more clear with the addition of the extra doors!

3. Back to number 1! 4 doors again, but this time, after you pick a door, Monty only opens ONE empty door (therefore, there are still 3 doors closed):



a. (**1 point**) What's the probability of winning if you stay with that door you picked? Hint: Don't overthink this! You'll know this answer from your computation of the answer in #1!

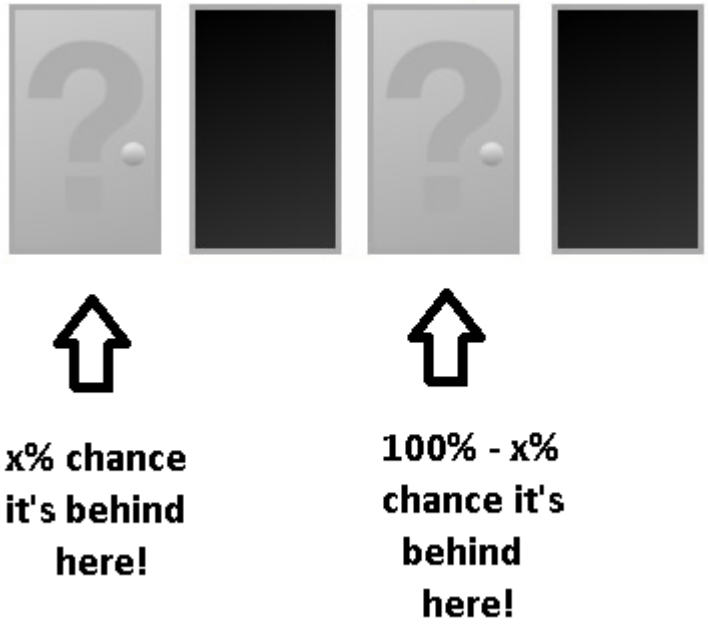
Make sure you check your result in 3a against this contingency table!

		If you pick and stay with…			
		Door #1	**Door #2**	**Door #3**	**Door #4**
…and the prize is actually behind…	**Door #1**	**…then you win! ☺**	**…then you lose. ☹**	**…then you lose. ☹**	**…then you lose. ☹**
	Door #2	**…then you lose. ☹**	**…then you win! ☺**	**…then you lose. ☹**	**…then you lose. ☹**
	Door #3	**…then you lose. ☹**	**…then you lose. ☹**	**…then you win! ☺**	**…then you lose. ☹**
	Door #4	**…then you lose. ☹**	**…then you lose. ☹**	**…then you lose. ☹**	**…then you win! ☺**

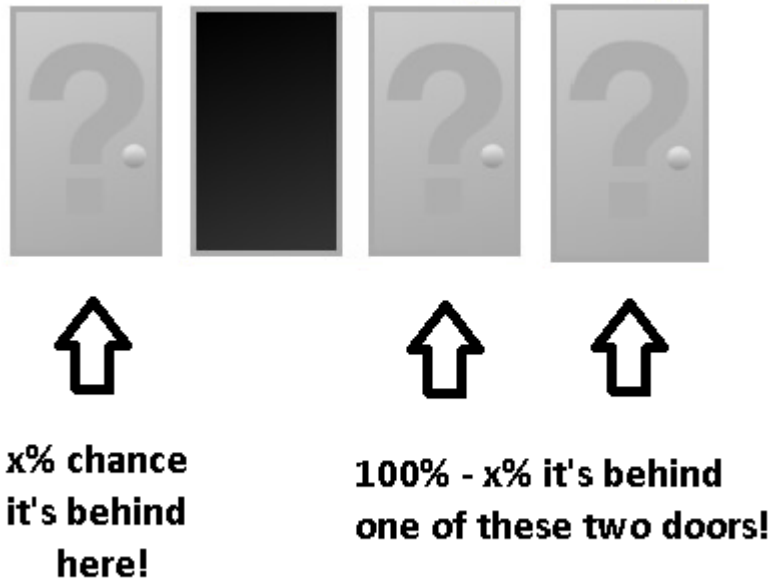
b. (**4 points**) What's the chance of winning if you *switch*? Careful! It's not as simple as subtracting the last one from 100%…remember, there are ***two*** doors to pick from if you switch, and only one can have the prize behind it! Let's analyze, shall we?

This one's going to require a slightly different attack – I don't think the contingency table approach will help. Let's attack it piece by piece!

- 100% of the time, the price is behind an unopened door, right? So that's why, in #1, you could just subtract the chance that it was behind the door you picked from 100%, and that was equal to the chance it was behind the other door:



- This time, a similar logic ensures! There's still a 100% chance that it's behind one of the remaining unopened doors:



- Now, you must ***guess*** at one of those two doors. You're just as likely to guess the door on the left as the door on the right, so it stands to reason that the probability that you got in question #1 needs to be split evenly between both doors. So, I ask again!

(**same 4 points!**) What's the chance of winning if you switch?

Quiz 3.

A few years back (too many to not make me feel really old) I stumbled upon the following fun math puzzler. It's called "The Prisoner problem", and it goes like this!

You are a prisoner sentenced to death. The emperor (who has sentenced you) offers you a chance to live by playing a simple game. He gives you 10 black marbles, 10 white marbles and 2 empty bowls. He then says, "Divide these 20 marbles into these 2 bowls. You can divide them any way you like as long as you place *all* the marbles *in* the bowls. Then I will blindfold you, and ask you to choose a bowl, and remove ONE marble from that bowl you have chosen.

You must leave the blindfold on the entire time, and must remove the first marble you touch. If the marble is white you will live, but if the marble is black, you will die."

So, to make this clear, there is absolutely ***no way*** to discern what color marble you have chosen, once you've selected it. There's no way to check a weight, or a sound, or anything else. Also, the emperor is cruel, but fair – he won't cheat you in any way.

1. (**1 point**) Let's shoot form the hip, for starters: what's your original gut feeling as to how you should arrange those marbles?

2. (**1 point**) Based on the arrangement you just gave, what is the chance that you will live? Remember, you can't game the emperor's system…you're just guessing at a bowl, and then randomly selecting a marble. What's the chance (given your arrangement) that you select a white marble?

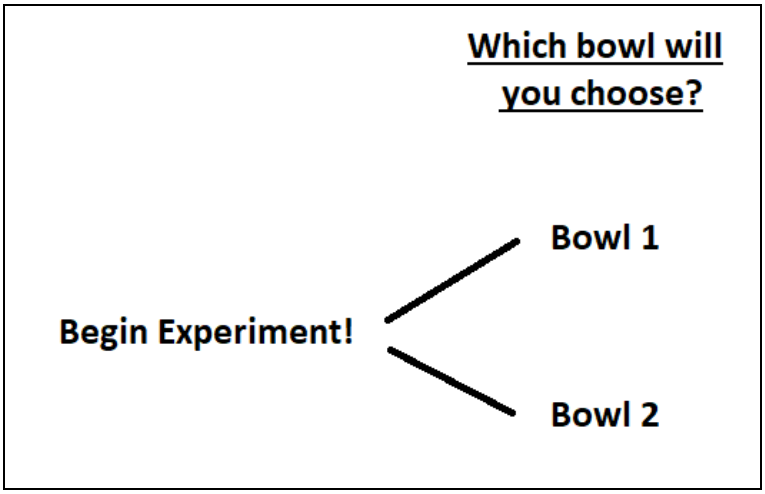
3. (**2 points**) OK! Now, the important part (for now!)…what does that probability in #2 ***mean***? In other words, if you said the chance of living was "30%"…what does that 30% ***mean***? 30% of ***what***?

OK! Now, from the years of my doing this problem with students, one of the most popular responses to #1 is (put 5 white and 5 black into both bowls". Then, to #2, they'll say, "You have a 50% chance to live." But then I ask them to show my why that's the case, mathematically. And that's what we'll do next!

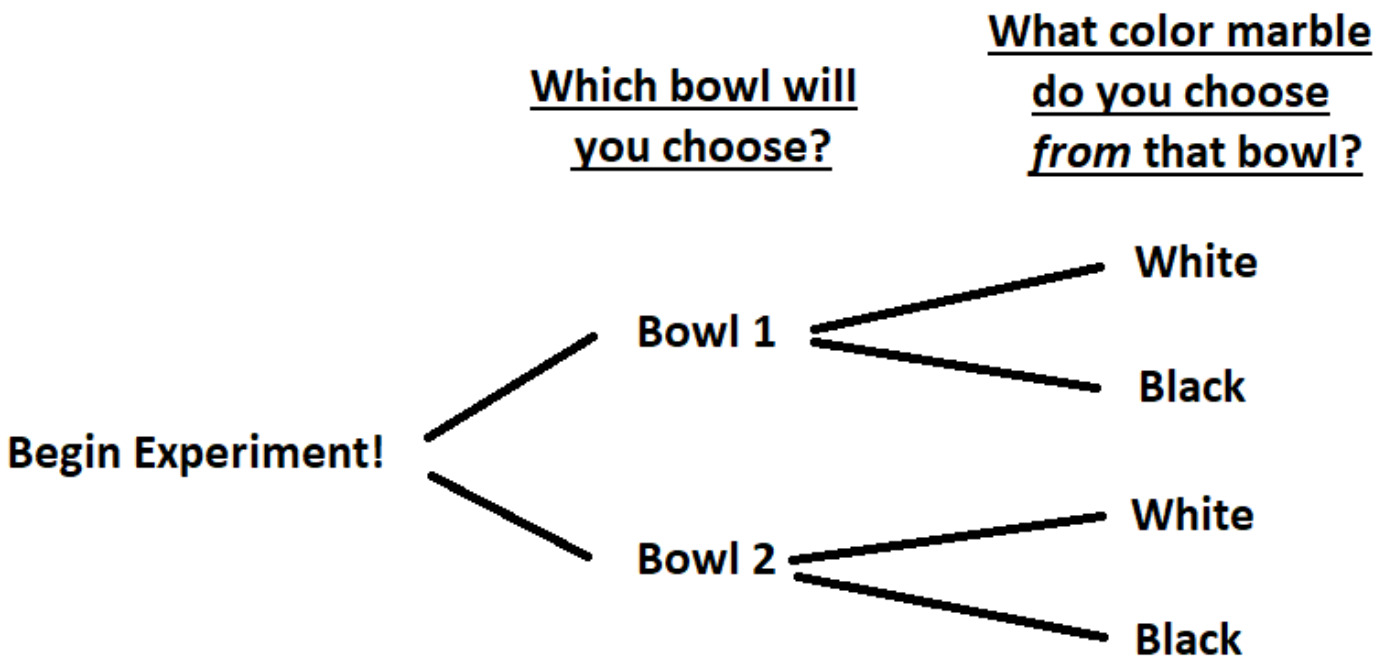
We may have used tree diagrams yet in class (maybe we haven't – they always pop up at some point, but I'm never sure when they will, term to term. ☺). If we haven't yet, we'll start right now!

A **tree diagram** is a pretty neat way to display all the outcomes of an experiment. In this case, the experiment is "select a bowl from one of two bowls, and then select a marble from that bowl". So, the first part of the experiment is "select a bowl from one of two bowls." In a tree diagram, that would look the one at right!

(I've labeled these bowls "Bowl 1" and "Bowl 2", but, in reality, were you doing this experiment, you wouldn't actually be able to differentiate between them. It helps, though, to label than to analyze it now).



Now that you've chosen a bowl, you need to choose a marble from that bowl. So here we go!



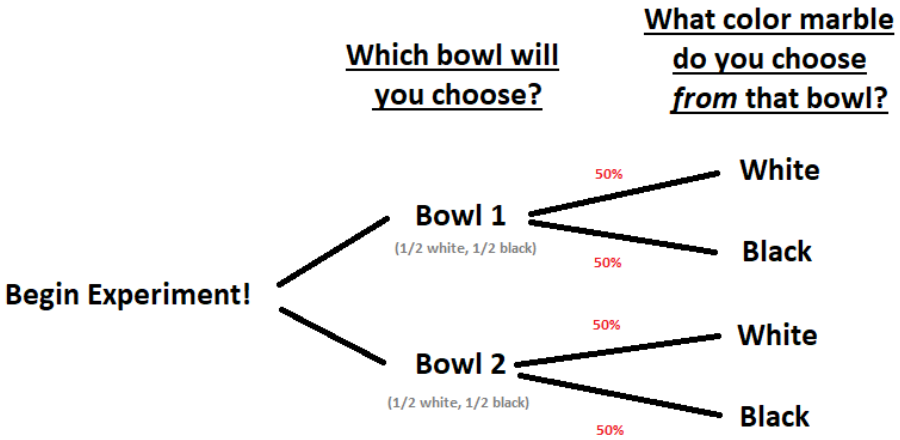
So, when you follow each branch of the tree out, you see all the possible ways this experiment can play out! Either a) you choose bowl 1 and get a white marble; b) you choose bowl 1 and get a black marble; c) you choose bowl 2 and get a white marble; or d) you choose bowl 2 and get a black marble. 100% of the time, one (and only one) of those things can happen if you do this experiment! So, mathematically:

Chance of getting bowl 1 and a white marble + Chance of getting bowl 1 and a black marble + Chance of getting bowl 2 and a white marble + Chance of getting bowl 2 and a black marble = 100%

Clearly, it's good if either the first or third of those scenarios play out (since you've selected white marbles)…so I'll color them **green** (for **go! You live**)!

Chance of getting bowl 1 and a white marble + Chance of getting bowl 1 and a black marble + Chance of getting bowl 2 and a white marble + Chance of getting bowl 2 and a black marble = 100%

The question now is: what do those two likelihoods add up to? According to the most common answer, it's "50%". But why? Let's return to the tree diagram for a moment…but add some information to it!

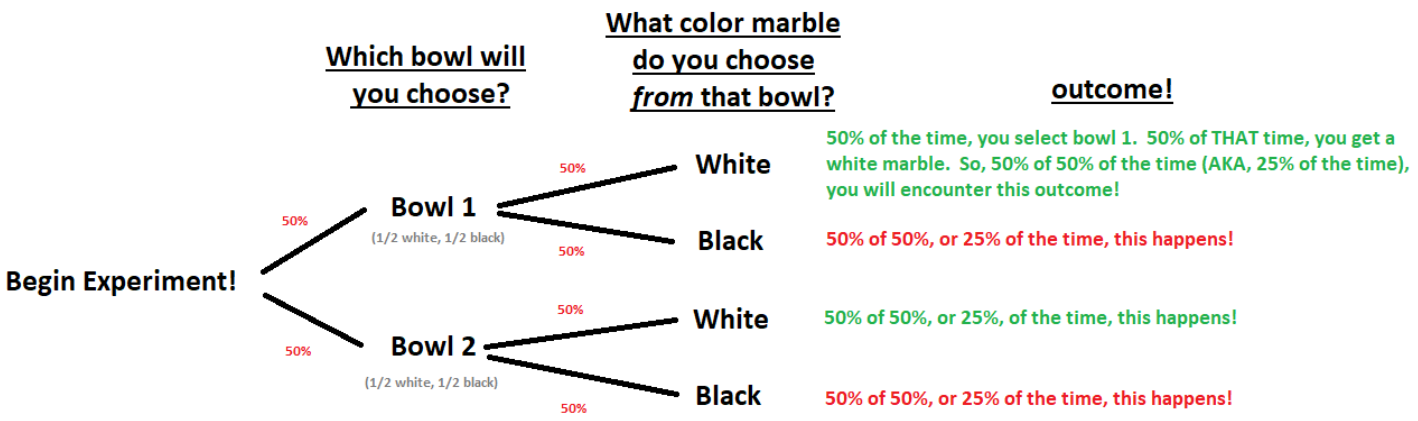


Let's make some sense of this one! In gray, under each bowl, I've just described how the marbles were arranged in those two bowls. The "50%", in red along each branch, indicate, therefore, the percentage of time that you will select either color of marble, once you have settled on certain bowl. So, if you pick bowl 1, there's a 50% chance that you will get a black marble (and a 50% chance you'll get a black one). Same for bowl #2. So, now we have it! Let's go back to the equation we developed above and add in some values!

Chance of getting bowl 1 and a white marble	+	Chance of getting bowl 1 and a black marble	+	Chance of getting bowl 2 and a white marble	+	Chance of getting bowl 2 and a black marble	=	100%
50%	+	50%	+	50%	+	50%	>	100%

Wait…***what***? How can 50% 4 times equal 100%? It equals ***200***%...and that makes ***NO*** sense!

There's something that we've ignored (and we're about to fix right now!)…sure, once you pick a bowl in this scenario, you're 50% likely to get a white marble. ***But, you only select that bowl half the time***! Remember – you're randomly selecting a bowl, ***before*** picking a marble. Stands to reason that half the prisoners would select bowl 1, and the others, bowl 2. Let's add some more info to the tree diagram!



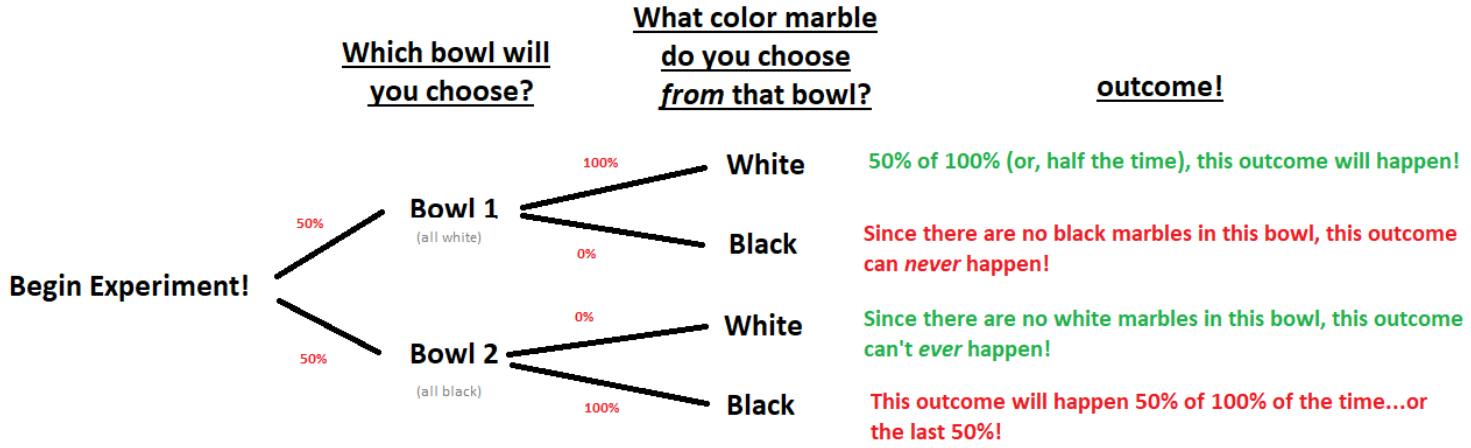
OK! This make a LOT more sense! Let's redo the equation!!!

Chance of getting bowl 1 and a white marble	+	Chance of getting bowl 1 and a black marble	+	Chance of getting bowl 2 and a white marble	+	Chance of getting bowl 2 and a black marble	=	100%
25%	+	25%	+	25%	+	25%	=	100%

And, adding the two green 25%'s together, we get 50%...which is the chance you will live under this scenario. Cool!

(If you like, you can test any arrangement of marbles using the spreadsheet attached to this quiz. Email me if you get stuck using it!)

Another common "shooting from the gut" solution for this is "put all the white marbles in one bowl, and all the black in the other". Let's look at ***that*** mathematically, as well!



Chance of getting bowl 1 and a white marble	+	Chance of getting bowl 1 and a black marble	+	Chance of getting bowl 2 and a white marble	+	Chance of getting bowl 2 and a black marble	=	100%
50%	+	0%	+	0%	+	50%	=	100%

So, just like the previous arrangement, the chance of winning is 50% if you arrange the marbles in an "all white, all black" way! And many students, after running through these two examples, tell me, "That's it You can't do better than 50%!"

To which I ask them, "Are you sure?" And then I ask them to test this arrangement: what if you put 9 white marbles in bowl 1, and the rest (that is, 10 black and 1 white) in bowl #2? What would your chance be ***then***?

4. (**2 points**) (**w**) Well, let's figure that out! Hint: it's better than 50% (not a ton better, but definitely ***better***)! Remember to show me how you got your answer!

At this point, students will often say, "Well, you never said you could put uneven numbers of marbles in the bowls!" To which I reply, "Go read the emperor's instructions again! Does he say you can't?" ☺

5. (**1 point**) (**w**) And so, we come to this: what would be the ***optimal*** arrangement of marbles, and what would the chance of your living with that arrangement be? Again, you can test your math with the spreadsheet!

Quiz 4.

In the last quiz, we attacked The Prisoner Problem (in case you've forgotten, just scroll on back up above this and check it out!)

1. (**1 points**) Through the quiz, we tried various scenarios until we maximized the chance of living. How did we do this? That is, describe the arrangement of the marbles in each of the bowls.

2. **(2 points)** (**w**) What ***is*** the maximal chance of living? Please round to the nearest hundredth of a percent (I know on the last quiz, I didn't specify this. It'll be important here!). Remember that the "(**w**)" means to show me how you got your result.

3. **(2 points)** (**w**) Now suppose that the emperor uses 100 black and 100 white marbles. Repeat the last question, using these new numbers – same rounding.

4. **(2 points)** (**w**) How about 1000 white and 1000 black?

5. **(1 point)** What percent does this appear to be approaching as the number of both colors of marbles increases at the same rate?

6. **(2 points)** Convince me, mathematically or otherwise, as to why the actual probability can never meet nor exceed your answer in **5**.

Quiz 5.

Let's re-revisit the prisoner problem again, in a new way:

You are a prisoner sentenced to death. The Emperor offers you a chance to live by playing a simple game. He gives you 10 black marbles, 10 white marbles and 3 empty bowls. He then says, "Divide these 20 marbles into these 3 bowls. You can divide them any way you like as long as you place all the marbles *in* the bowls (and each bowl needs to have at least one marble in it). Then I will blindfold you and mix the bowls around. You then can choose one bowl and remove ONE marble. You must leave the blindfold on the entire time, and must remove the first marble you touch. If the marble is white you will live, but if the marble is black, you will die."

Show each calculation you used like you did on the last quiz!

1. **(2 points)** (**w**) What is the maximal chance of living? Please round to the nearest tenth of a percent.

2. **(2 points)** (**w**) Let's change it again! This time, 10 white marbles, 10 black marbles, but you have **10** bowls.

3. **(6 points)** (**w**) One last change - 10 white marbles, 10 black marbles, and ***11*** bowls. Please explain all strategies you tried, in order, including those that didn't work – problem solving is about refinement!