Full Derivation of the Centripetal Acceleration
# (No short-cuts)

Required skills required to follow the derivation include:
- Cartesian vs. Polar coordinates
- Adding vector components
- Derivatives
- Product Rule
- Chain Rule
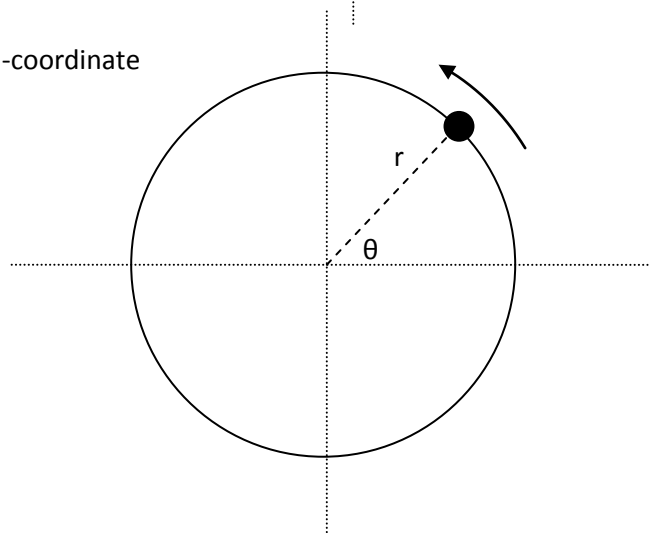- Implicit Differentiation
- Derivatives of sine and cosine

The following derivation will be very confusing for any student who has not <u>completed</u> derivatives.

Establishing rotation in rectangular and polar coordinates:
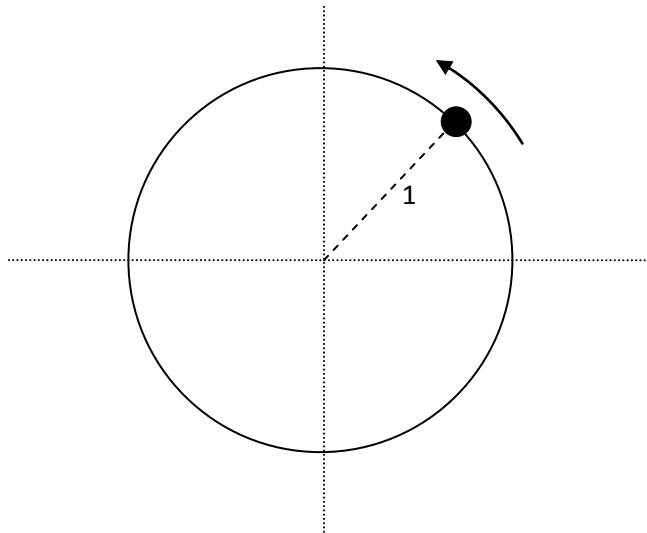
Imagine an object moving in a perfect circle…
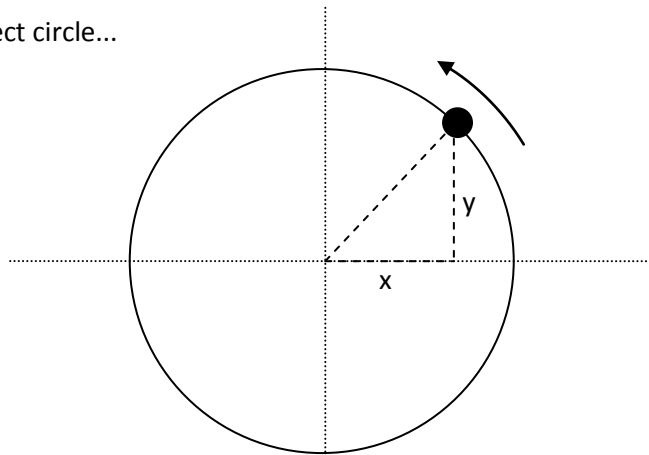
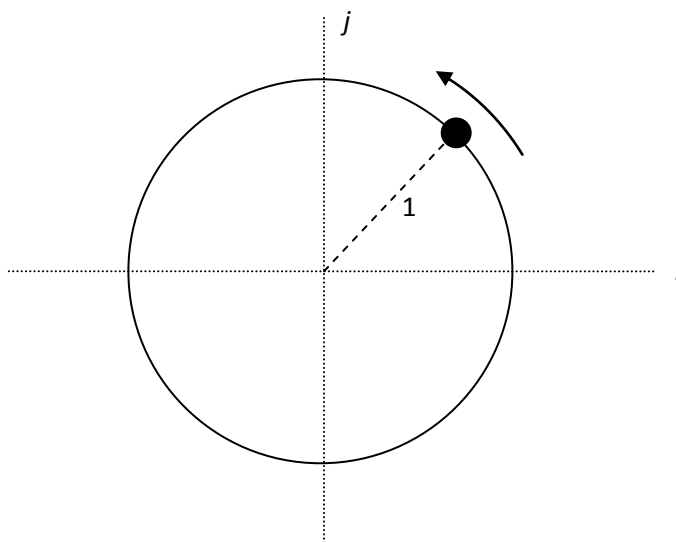The object is always at some (x,y)-coordinate

Or we can say the object is at a (r, θ) coordinate where r is the radial distance from the center of the circle and θ is the angle shown above.
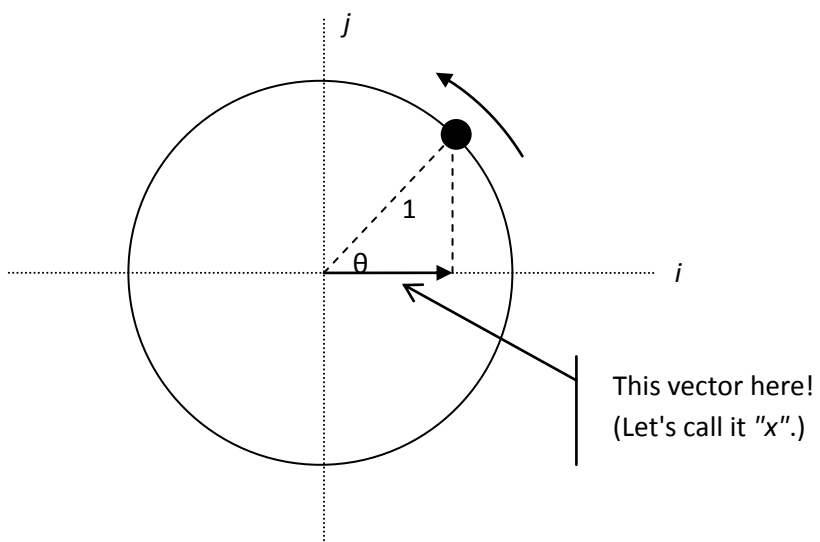
Let's give the circle a radius of 1:

And let's call the x-direction i, and the y-direction j.

So, what is the vector shown below?

This vector here!
(Let's call it "x".)

$$\cos \theta = \frac{adjacent}{hypotenuse} = \frac{x}{1}$$

$$x = 1 \cdot \cos \theta$$

$$x = \cos \theta \; \hat{\imath}$$

The $\hat{\imath}$ simply means that the vector points in the *i*-direction.

What about the new vector shown below?

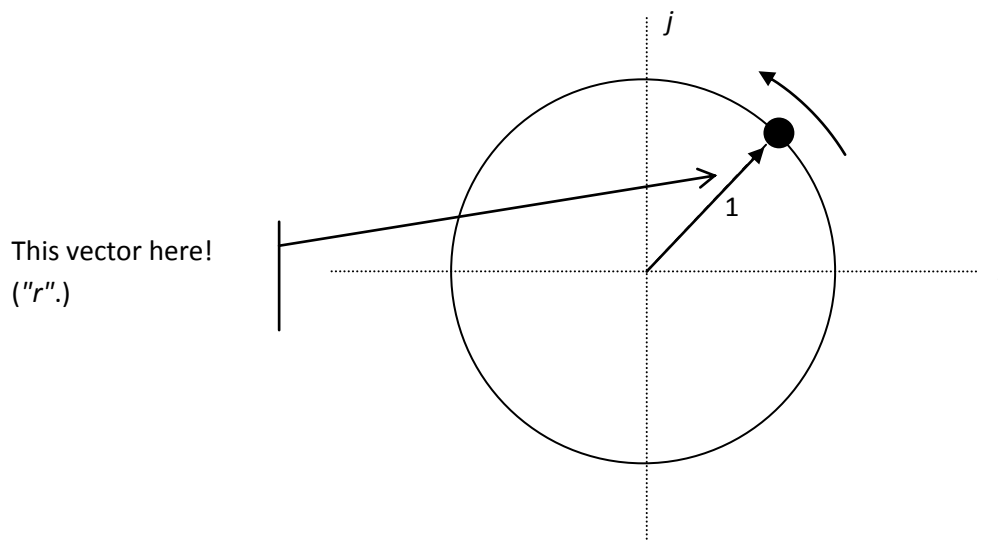
This vector here!
(Let's call it "y".)
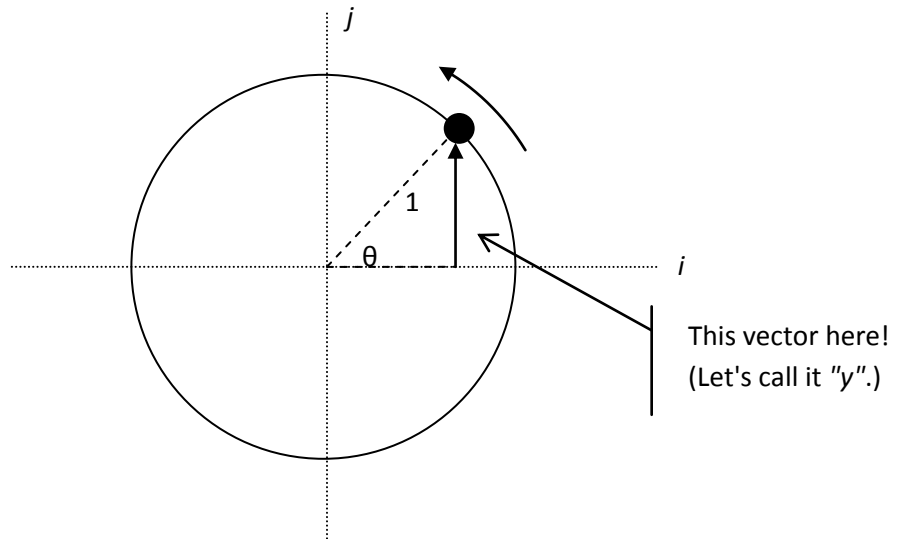
$$\sin \theta = \frac{opposite}{hypotenuse} = \frac{y}{1}$$

$$y = 1 \cdot \sin \theta$$

$$y = \sin \theta \; \hat{\jmath}$$

The $\hat{\jmath}$ simply means that the vector points in the *j*-direction.

So...what is **this** vector?

This vector here!
("r".)

If you go sideways by the x-vector, and up by way of the y-vector, you get the same result as the r-vector. You can say this with math by writing...
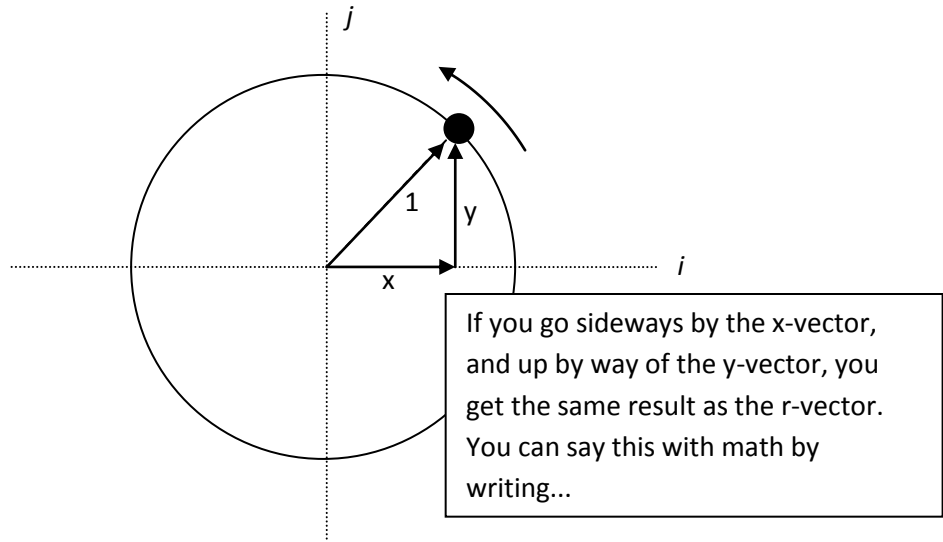
$$\hat{r} = \vec{x} + \vec{y}$$

$$\boldsymbol{\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta}$$

Now to figure out what $\hat{\theta}$ is in terms of $\hat{i}$ and $\hat{j}$.

The first thing to determine is which way the $\hat{\theta}$ is pointing. The $\hat{\theta}$ is the angle opening up as the object revolves around the circle. It always points perpendicular to $\hat{r}$ as is shown above.

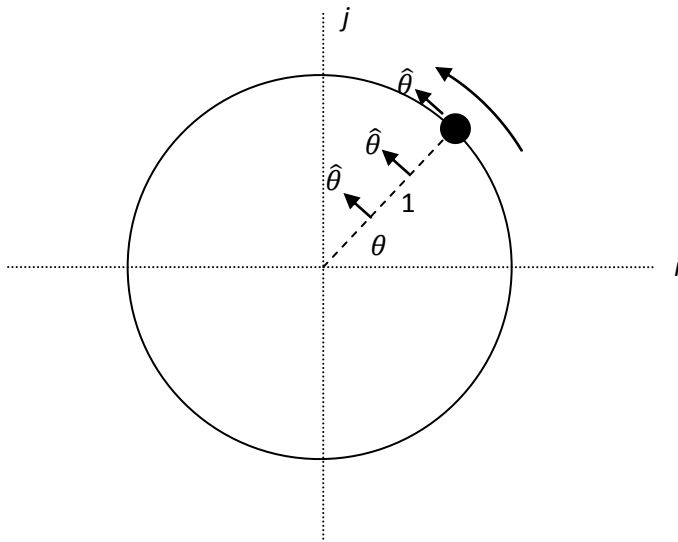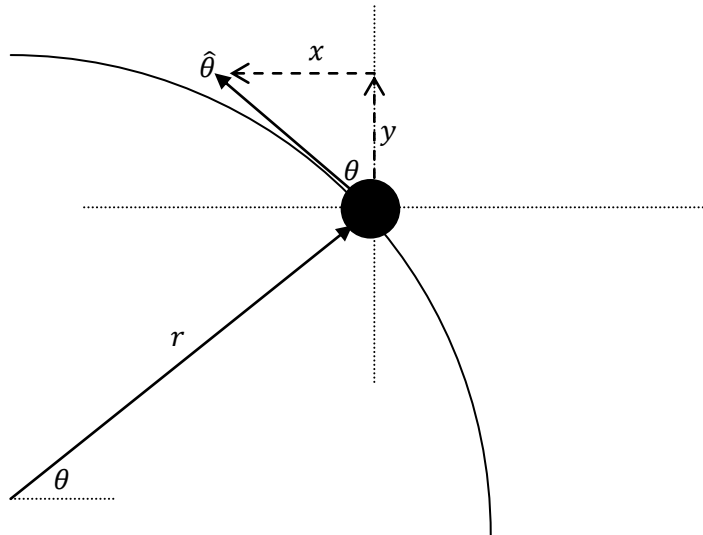Let's zoom in on the top $\hat{\theta}$ in the diagram above:

We can use the same logic we used before to figure out x and y, using sine and cosine. Notice however that the x is now opposite and the y is now adjacent.

So instead of:

$$\hat{\theta} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$ Not Correct!

We will get:

$$\hat{\theta} = \hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$ Still not quite correct!

Keep in mind that the x-vector points to the left now instead of to the right...

$$\boldsymbol{\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta}$$ Finally correct!

Taking the derivative of $\hat{r}$ and $\hat{\theta}$ so we have them for later:

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Let's take the derivative of each one with respect to time and see what we get ...

First, the derivative of $\hat{r}$:

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta) = \left[-\hat{\imath}\sin\theta\left(\frac{d\theta}{dt}\right) + \hat{\jmath}\cos\theta\left(\frac{d\theta}{dt}\right)\right]$$

... by using the chain rule

$$= (-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)\left(\frac{d\theta}{dt}\right)$$

... by distributing out the $\dfrac{d\theta}{dt}$

$$\boldsymbol{\frac{d\hat{r}}{dt} = \hat{\theta}\left(\frac{d\theta}{dt}\right)}$$

... by realizing that $-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$ was our original definition of $\hat{\theta}$

Now, the derivative of $\hat{\theta}$:

$$\frac{d\hat{\theta}}{dt} = \frac{d}{dt}(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta) = \left[-\hat{\imath}\cos\theta\left(\frac{d\theta}{dt}\right) - \hat{\jmath}\sin\theta\left(\frac{d\theta}{dt}\right)\right]$$

... by using the chain rule

$$(-\hat{\imath}\cos\theta - \hat{\jmath}\sin\theta)\left(\frac{d\theta}{dt}\right)$$

... by distributing out the $\dfrac{d\theta}{dt}$

$$\boldsymbol{\frac{d\hat{\theta}}{dt} = -\hat{r}\left(\frac{d\theta}{dt}\right)}$$

... by realizing that $\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$ was our original definition of $\hat{r}$

Finding Centripetal Acceleration:

$$position = \vec{r}$$

$$position = r \cdot \hat{r}$$

$$velocity = time\ derivative\ of\ position = \frac{d\vec{r}}{dt}$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r \cdot \hat{r})$$

using the product rule ...

$$\frac{d}{dt}(r \cdot \hat{r}) = r\left(\frac{d\hat{r}}{dt}\right) + \hat{r}\left(\frac{dr}{dt}\right)$$

Remember from before: $\dfrac{d\hat{r}}{dt} = \hat{\theta}\left(\dfrac{d\theta}{dt}\right)$

So, now ...

$$\frac{d\vec{r}}{dt} = r\hat{\theta}\left(\frac{d\theta}{dt}\right) + \hat{r}\left(\frac{dr}{dt}\right)$$

The derivative of position with respect to time is the definition of velocity.

$$\frac{d\vec{r}}{dt} = r\hat{\theta}\left(\frac{d\theta}{dt}\right) + \hat{r}\left(\frac{dr}{dt}\right) = \vec{v}$$

... and the derivative of velocity with respect to time is the acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[r\hat{\theta}\left(\frac{d\theta}{dt}\right) + \hat{r}\left(\frac{dr}{dt}\right)\right]$$

Now, to do a lot of product rules ...

We will do the last term first, since it is easier ...

$$\frac{d}{dt}\hat{r}\left(\frac{dr}{dt}\right) = \hat{r}\left(\frac{d^2r}{dt^2}\right) + \left(\frac{dr}{dt}\right)\frac{d\hat{r}}{dt}$$

*Remember **again** that* $\dfrac{d\hat{r}}{dt} = \hat{\theta}\left(\dfrac{d\theta}{dt}\right)$

So ...

$$\frac{d}{dt}\hat{r}\left(\frac{dr}{dt}\right) = \hat{r}\left(\frac{d^2r}{dt^2}\right) + \left(\frac{dr}{dt}\right)\hat{\theta}\left(\frac{d\theta}{dt}\right) \qquad \boxed{\text{Part 1}}$$

Now for the more difficult, first term ...

$$\frac{d}{dt}\left[r\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] = r\frac{d}{dt}\left(\left(\frac{d\theta}{dt}\right)\right)\hat{\theta} + \left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)\hat{\theta} + \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\left(\frac{d\hat{\theta}}{dt}\right)$$

First part of product rule

Second part of product rule

Chain rule for the $\hat{\theta}$

Again, the $\dfrac{d\hat{\theta}}{dt} = -\hat{r}\left(\dfrac{d\theta}{dt}\right)$

$$\frac{d}{dt}\left[r\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] = r\frac{d}{dt}\left(\left(\frac{d\theta}{dt}\right)\right)\hat{\theta} + \left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)\hat{\theta} + (r)\left(\frac{d\theta}{dt}\right)\left(-\hat{r}\left(\frac{d\theta}{dt}\right)\right) \qquad \boxed{\text{Part 2}}$$

$$a = \left[\frac{d^2r}{dt^2} \cdot \hat{r} + \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] + \left[r\frac{d^2\theta}{dt^2} \cdot \hat{\theta} + \left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)\hat{\theta} - (r)\left(\frac{d\theta}{dt}\right)^2\hat{r}\right]$$

Simplifying the acceleration down:

If something is moving in a circle, then the radius is never changing. This makes $\frac{dr}{dt} = 0$, *and* $\frac{d^2r}{dt^2} = 0$.

$$a = \left[\frac{d^2r}{dt^2} \cdot \hat{r} + \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] + \left[r\frac{d^2\theta}{dt^2} \cdot \hat{\theta} + \left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)\hat{\theta} - (r)\left(\frac{d\theta}{dt}\right)^2\hat{r}\right]$$

$$a = \left[0 \cdot \hat{r} + (0)\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] + \left[r\frac{d^2\theta}{dt^2} \cdot \hat{\theta} + \left(\frac{d\theta}{dt}\right)(0)\hat{\theta} - (r)\left(\frac{d\theta}{dt}\right)^2\hat{r}\right]$$

$$a = \left[r\frac{d^2\theta}{dt^2} \cdot \hat{\theta} - (r)\left(\frac{d\theta}{dt}\right)^2\hat{r}\right]$$

And because r is perpendicular to $\hat{\theta}$, $\quad r \cdot \hat{\theta} = 0$

$$a = \left[0 - (r)\left(\frac{d\theta}{dt}\right)^2\hat{r}\right]$$

$$\boldsymbol{a = -r\left(\frac{d\theta}{dt}\right)^2 \hat{r}}$$

Whatever $r\left(\frac{d\theta}{dt}\right)^2$ *is we will figure out next. The important part is that the whole thing is followed by* $\hat{r}$, *meaning that the direction is either out from the center or in towards the center. The negative means the direction is inwards instead of outwards. This proves that the direction of acceleration is always towards the center of the circle.*
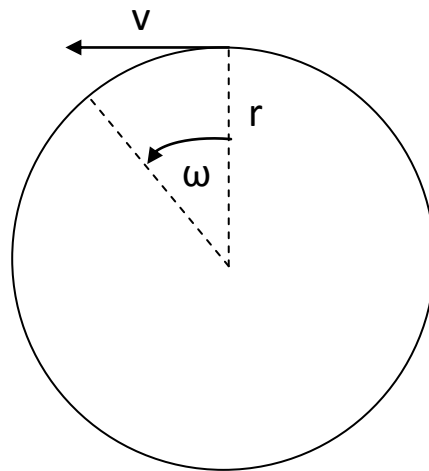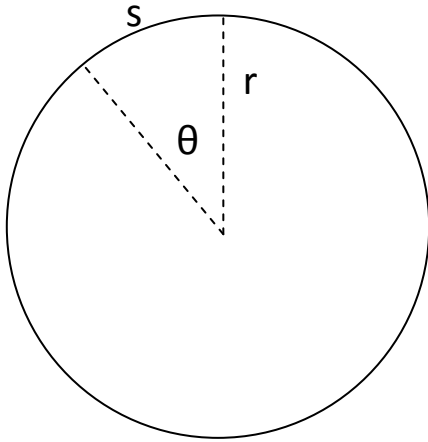
Moving back from polar to cartesian:

$$a = -r\left(\frac{d\theta}{dt}\right)^2 \hat{r}$$

θ *is the angular position, so* $\frac{d\theta}{dt}$ *is the angular velocity, also known as* ω *…*

$$a = -r(\omega)^2\hat{r}$$

Just like $\theta = \frac{s}{r}$, $\omega = \frac{v}{r}$. *Where* v *is the tangential velocity.*

$$a = -r\left(\frac{v}{r}\right)^2 \hat{r}$$

$$a = -\frac{v^2}{r}\hat{r}$$

…Which is the exact equation for centripetal acceleration. Again, the $\hat{r}$ simply means the direction is radial and the negative is just there to show inwards as opposed to outwards.