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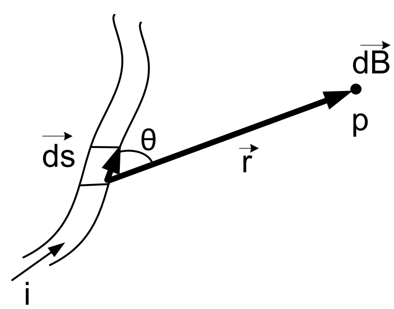
Magnetic Fields Due to Currents

Objectives:

1. To study Biot-Savart law.
2. To study Ampere's law.
3. To learn how to compute the magnetic field for a current-carrying wire.
4. To learn how to compute the force between two parallel current-carrying wires.

Biot–Savart Law

Biot and Savart found that the magnitude of the magnetic field contribution (**dB**) is directly proportional to the amount of current (**i**) and the length of the small segment of wire (**ds**).They also found that the magnitude of the magnetic field at a point (**P**) decreases as the inverse square of the distance between the segment of the wire and point **P**.



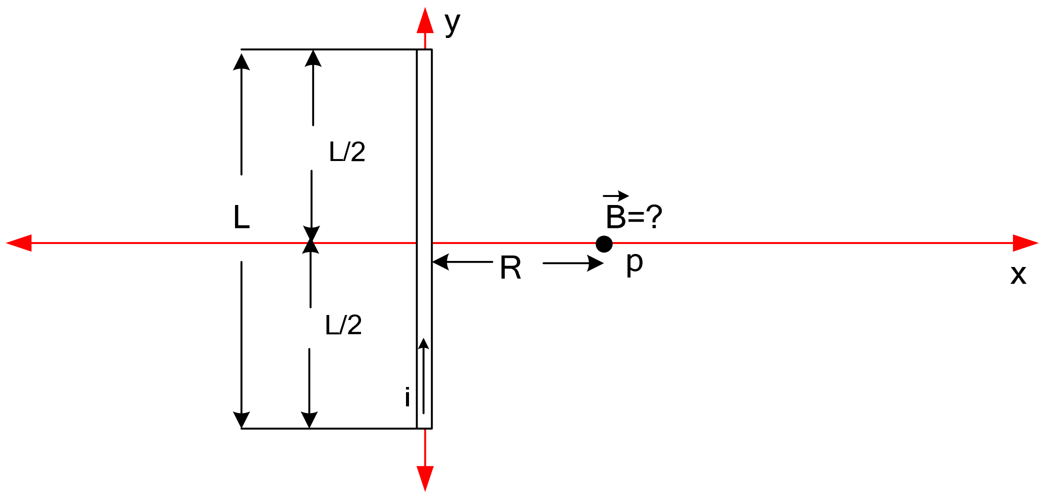
$$|dB| = \frac{\mu_0 \, i \, ds \, sin(\theta)}{4\pi \;\, r^2}$$

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi} \frac{i \, \overrightarrow{ds} \times \vec{r}}{r^3} (Biot - Savart'sLaw)$$

Example:#1

In the following figure, a straight wire of length **L** carrying current **i**.

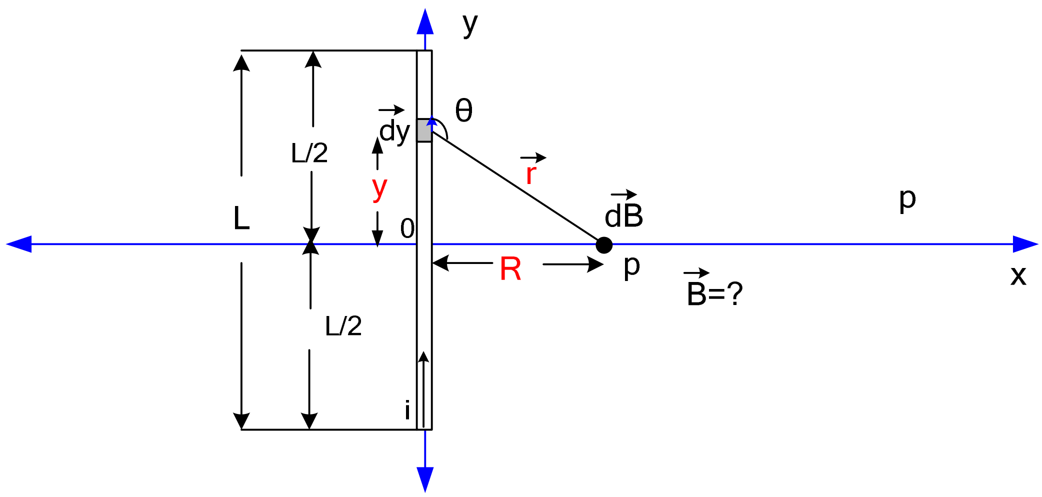
(a) Use Biot-Savart law to find $\vec{B}$ at a distance **R** from the segment along the perpendicular bisector.
(b) Find $\vec{B}$ if the wire is very long (infinite length).



Solution:

For a small segment (dy) of the wire:

$$dB = \frac{\mu_0 \, i \, dy \, \sin(\theta)}{4\pi \, r^2} \dots\dots\dots (1)$$



It is straightforward to show that:

$$r = \sqrt[2]{y^2 + R^2}$$

$$sin(\theta) = \sin(180 - \theta) = \frac{R}{\sqrt[2]{y^2 + R^2}}$$

Substitute r and $sin(\theta)$ in Equation (1):

$$dB = \frac{\mu_0\ i\ dy\ \ \frac{R}{\sqrt[2]{y^2 + R^2}}}{4\pi\ \ (\sqrt[2]{y^2 + R^2})^2} = \frac{\mu_0\ i\ R\ dy}{4\pi\ \ (y^2 + R^2)^{3/2}}$$

Due to the symmetry:

$$B = 2\int_0^{L/2} dB$$

$$B = 2\int_0^{L/2} \frac{\mu_0\ i\ R\ dy}{4\pi\ \ (y^2 + R^2)^{3/2}} = \frac{\mu_0\ i\ R}{2\pi}\int_0^{L/2} \frac{dy}{(y^2 + R^2)^{3/2}} \ldots\ldots(2)$$

Use the following trigonometric substitution:

$$y = R\tan(u)$$

Then

$$dy = R\frac{1}{\cos^2(\mathrm{u})}\ du$$

In Eq. 2

$$\frac{\mu_0\ i\ R}{2\pi}\int_{y=0}^{y=\frac{L}{2}} \frac{R\frac{1}{\cos^2(\mathrm{u})}\ du}{(R^2\tan^2(u) + R^2)^{\frac{3}{2}}} =$$

$$= \frac{\mu_0\ i\ R}{2\pi}\int_{y=0}^{y=L/2} \frac{R\frac{1}{\cos^2(\mathrm{u})}\ du}{(R^2\frac{1}{\cos^2(\mathrm{u})})^{3/2}}$$

$$= \frac{\mu_0\ i}{2\pi\ R}\int_{y=0}^{y=\frac{L}{2}} \cos(u)\ du = \frac{\mu_0\ i}{2\pi\ R}\ \ [\sin(u)]_{\mathrm{y=0}}^{\mathrm{y=L/2}}$$

Since y= $R\tan(u)$ → $\sin(u) = \frac{y}{\sqrt[2]{y^2+R^2}}$

Then,

$$\frac{\mu_0\, i}{2\pi\, R}\ [\sin(u)]_{y=0}^{y=L/2} = \frac{\mu_0\, i}{2\pi\, R}\left[\frac{y}{\sqrt[2]{y^2+R^2}}\right]_0^{L/2}$$

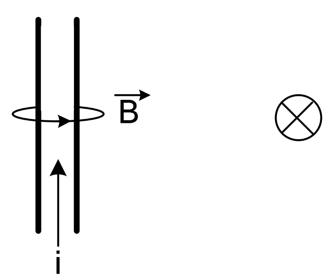
$$= \frac{\mu_0\, i}{2\pi\, R}\frac{(L/2)}{\sqrt{(\frac{L}{2})^2+R^2}} = \frac{\mu_0\, i}{2\pi\, R}\frac{L}{\sqrt{L^2+4\,R^2}}$$

$$B = \frac{\mu_0\, i}{2\pi R}\frac{L}{\sqrt{L^2+4\,R^2}} \quad (in\ the\ middle\ at\ a\ distance\ R)$$

b) For a wire of **infinite length** (long wire: L>>R)

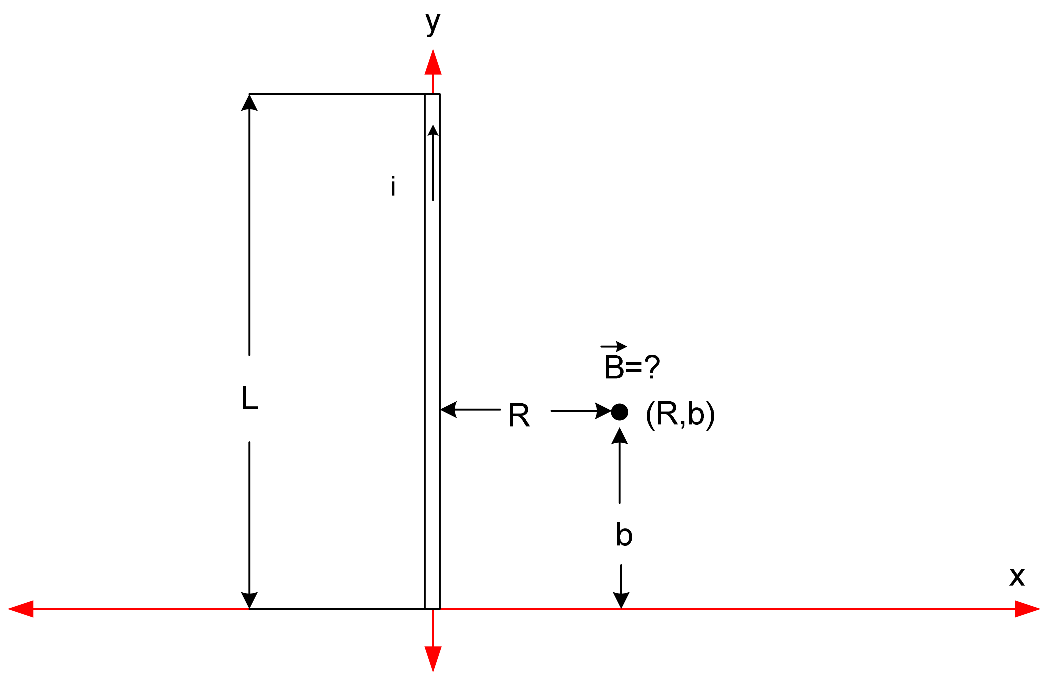
$$B = \frac{\mu_0\, i}{2\pi R} \qquad (for\ a\ very\ long\ wire)$$

Using the right-hand rule the direction of B is into the page



Example #2:

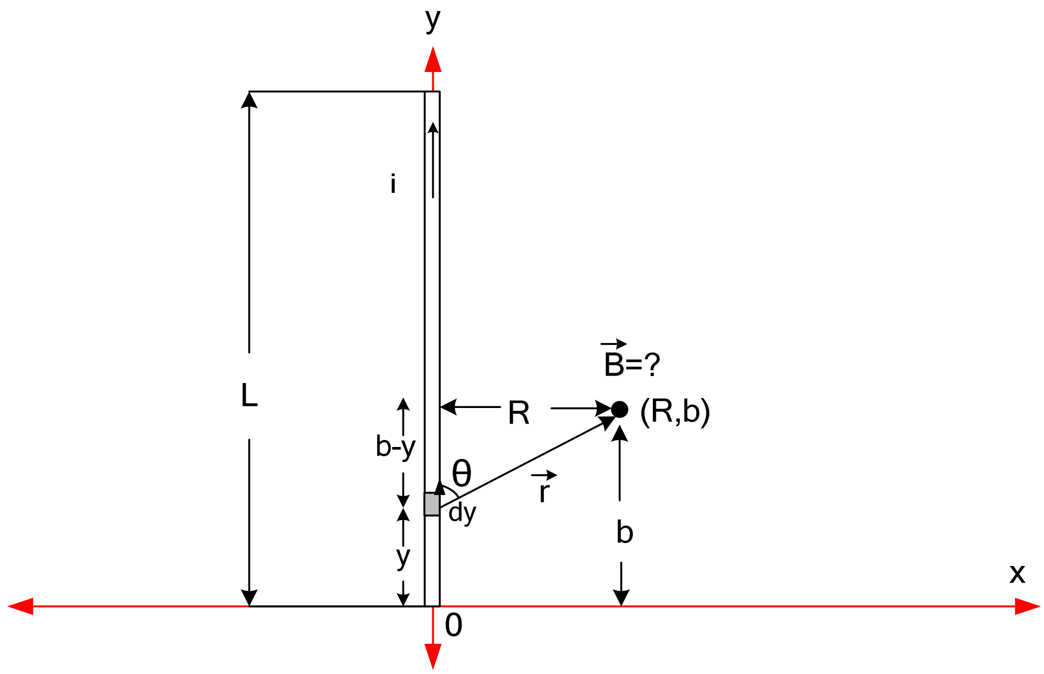
Find the magnetic field $\vec{B}$ at a point P →(R,b)



Solution:

For a small segment (dy) of the wire:

$$dB = \frac{\mu_0 \, i \, dy \; sin(\theta)}{4\pi \; r^2} \ldots \ldots \ldots (3)$$



It is straightforward to show that:

$$r = \sqrt[2]{(b-y)^2 + R^2}$$

$$sin(\theta) = \frac{R}{\sqrt[2]{(b-y)^2 + R^2}}$$

Then Equation (3) becomes:

$$dB = \frac{\mu_0 \, i \, dy \;\; \frac{R}{\sqrt[2]{(b-y)^2 + R^2}}}{4\pi \;\; (\sqrt[2]{(b-y)^2 + R^2})^2} = \frac{\mu_0 \, i \, R \, dy}{4\pi \;\; ((b-y)^2 + R^2)^{3/2}}$$

There is NO symmetry:

$$B = \int_0^L dB$$

$$B = \int_0^L \frac{\mu_0 \, i \, R \, dy}{4\pi \;\; ((b-y)^2 + R^2)^{3/2}} = \frac{\mu_0 \;\; i \, R}{4\pi} \int_0^L \frac{dy}{((b-y)^2 + R^2)^{3/2}}$$

Substitute u= $b - y$, then dy=-du

if y=L → u=b-L

if y=0 → u=b

$$B = \frac{\mu_0\, i\, R}{4\pi} \int_{b}^{b-L} \frac{-du}{(u^2 + R^2)^{3/2}}$$

$$B = \frac{\mu_0\, i\, R}{4\pi} \int_{b-L}^{b} \frac{du}{(u^2 + R^2)^{3/2}}$$

$$B = \frac{\mu_0\, i\, R}{4\pi} \left[\frac{u}{R^2\, \sqrt[2]{u^2 + R^2}}\right]_{b-L}^{b}$$

$$B = \frac{\mu_0\, i\, R}{4\pi\, R^2} \left[\frac{b}{\sqrt[2]{b^2 + R^2}} - \frac{b-L}{\sqrt[2]{(b-L)^2 + R^2}}\right]$$

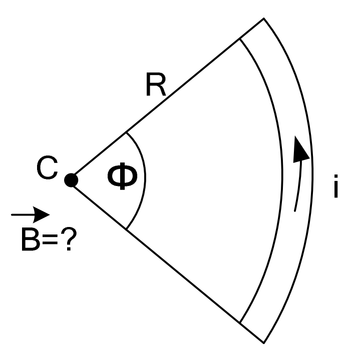
$$\boxed{B = \frac{\mu_0\, i}{4\pi\, R} \left[\frac{b}{\sqrt[2]{b^2 + R^2}} - \frac{b-L}{\sqrt[2]{(b-L)^2 + R^2}}\right]}$$

Remark:

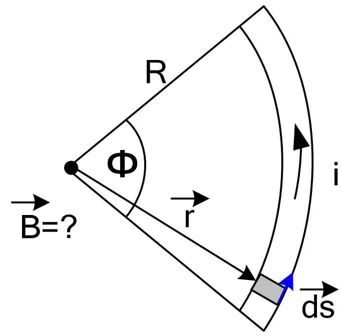
If **b=L/2,** then we get the same result of the previous example. (Do it by yourself)

Example #3

Find the magnetic field ($\vec{B}$) at C due to current-length element in a circular arc; as show in the figure below:



Solution:



$$dB = \frac{\mu_0}{4\pi} \frac{i\,\vec{ds} \times \vec{r}}{r^3}$$

We notice that $\vec{ds}$ is always perpendicular to $\vec{r}$ → $\theta = 90$→ sin(90) $= 1$.

Then,

$$dB = \frac{\mu_0}{4\pi} \frac{i\,ds}{r^2}$$

Use the following identity to change the integration from ds to $d\emptyset$

$$ds = R\,d\emptyset$$

Then,

$$dB = \frac{\mu_0}{4\pi} \frac{i\,R\,d\emptyset}{r^2}$$

$$B = \int \frac{\mu_0}{4\pi} \frac{i\,R\,d\emptyset}{r^2} = \frac{\mu_0}{4\pi} \frac{i\,R}{r^2} \int_0^{\emptyset} d\emptyset$$

$$B = \frac{\mu_0}{4\pi} \frac{i\,R\,\emptyset}{r^2}$$

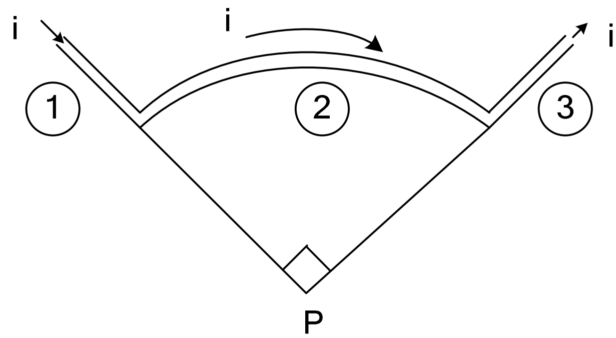
As we can see that r is always equal to R then:

$$\boxed{B = \frac{\mu_0}{4\pi} \frac{i\ \emptyset}{R}}$$

Note: $\emptyset$: $in\ radians\ NOT\ degrees$

Example #4

For the following figure, find $\vec{B}$ at P.



Solution:

The wire can be divided into three segments:

B1=0, because the angle θ between ds and r is equal to 0; (sin(0)=0)

B3=0, because the angle θ between ds and r is equal to 180; (sin(180)=0)

B2 can be obtained using the previous example:

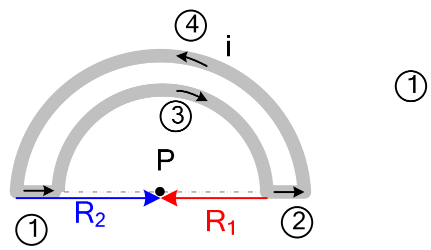
$$B_2 = \frac{\mu_0}{4\pi} \frac{i \ \emptyset}{R} = \frac{\mu_0}{4\pi} \frac{i \ (\pi/2)}{R} = \frac{\mu_0 \, i}{8 \, R}$$

$$B_{net} = B_1 + B_2 + B_3 = \frac{\mu_0 \, i}{8 \, R}$$

The direction is into the page (use the right-hand rule).

Example #5

Find the magnetic filed $\vec{B}$ at P.



Solution

The wire can be divided into four segments:

B1=0, because the angle θ between ds and r is equal to 0; (sin(0)=0)

B2=0, because the angle θ between ds and r is equal to 180; (sin(0)=0)

$$B_{\mathbf{3}} = \frac{\mu_0}{4\pi}\frac{i\ \emptyset}{R_1} = \frac{\mu_0}{4\pi}\frac{i\ \pi}{R_1} = \frac{\mu_0}{4}\frac{i}{R_1}$$

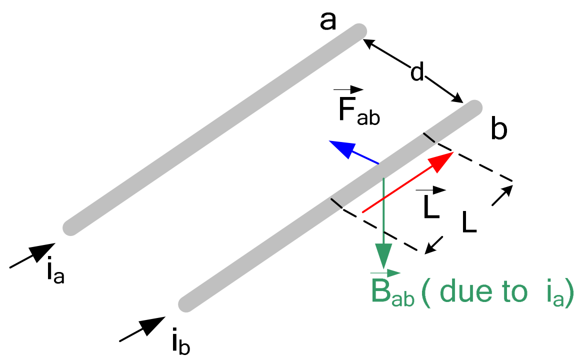
$$B_{\mathbf{4}} = \frac{\mu_0}{4\pi}\frac{i\ \emptyset}{R_2} = \frac{\mu_0}{4\pi}\frac{i\ \pi}{R_2} = \frac{\mu_0}{4}\frac{i}{R_2}$$

$$B_{\boldsymbol{net}} = \frac{\mu_0\, i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

If B is into the page (+)

If B is out of the page (-)

Force Between Two Parallel Currents



The magnetic filed due to i_a:

$$B_{\boldsymbol{ab}} = \frac{\mu_0 \, |i_a|}{2\pi d}$$

The expression of the force on length of current-carrying wire tells us that the force on wire **b** is:

$$\overrightarrow{F_{\boldsymbol{ab}}} = i_b \, \vec{L} \, \text{x} \, \overrightarrow{B_{\boldsymbol{ab}}}$$

$$F_{\boldsymbol{ab}} = |i_b| \, L \;\; B_{\boldsymbol{ab}} \, sin(90) = \frac{\mu_0 \, L \, |i_a \, i_b|}{2\pi d}$$

The direction of $\overrightarrow{F_{ab}}$ is in the direction of the cross product $\vec{L} \, \text{x} \, \overrightarrow{B_{ab}}$. Applying the right-hand rule, we find that $\overrightarrow{F_{ab}}$ points directly toward wire **a** as shown.