Chapter 1

The Wonderful World of Differential Equations

> The essential fact is simply that all the pictures which science now draws of nature, and which alone seem capable of according with observational fact, are mathematical pictures. ...
>
> Sir James Jeans[1]

1.1 What are Differential Equations?

Just what are differential equations? Following the wisdom of the old Chinese proverb that "one picture is worth more than a thousand words," we defer our answer until we have provided a picture of sorts. Table 1.1 is, so to speak, a collage of various types of differential equations. With one exception, these are well-known equations drawn from different scientific and technical disciplines. A sense of their importance may be realized from their ability to mathematically describe, or model, real-life situations. The equations come from the diverse disciplines of demography, ecology, chemical kinetics, architecture, physics, mechanical engineering, quantum mechanics, electrical engineering, civil engineering, meteorology, and a relatively new science called *chaos*. The same differential equation may be important to several disciplines, although for different reasons. For example, demographers, ecologists, and mathematical biologists would immediately recognize

$$\frac{dp}{dt} = rp,$$

the first equation in Table 1.1, as the *Malthusian law of population growth*. It is used to predict populations of certain kinds of organisms reproducing under ideal conditions—

[1]See [**?**, ch. 5]. Sir James Jeans (1877–1946) was a British mathematical physicist, Cambridge University lecturer, Princeton University professor of applied mathematics, and author of a number of popular works of science, of which *The Mysterious Universe* [**?**] was one of his most famous. His treatise, *Problems of Cosmogony and Stellar Dynamics* (1917), on the behavior of fluids in space contributed to a greater understanding of the origin and evolution of the universe.

whereas physicists, chemists, and nuclear engineers would be more inclined to regard the equation as a mathematical portrayal, or model, of radioactive decay. Even many economists and mathematically minded investors would recognize this differential equation, but in a totally different context: it also models future balances of investments earning interest at rates compounded continuously.

Another example is the *van der Pol equation*

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0,$$

which came from modeling oscillations of currents in the nonlinear electrical circuits of the first commercial radios. For many years, it was the subject of research by electrical engineers and mathematicians alike.

Table 1.1: **Differential Equations Modeling Real-Life Situations**

Differential Equation	**Situation**
$\frac{dp}{dt} = rp$	The *Malthusian law of population growth* is used to model the populations of certain kinds of organisms living in ideal environments for limited lengths of time t. It gives the rate at which a population p changes with respect to t. The value of the constant r depends on the organism.
$\frac{dx}{dt} = k(A - x)^2$	This *second-order reaction rate* law gives the rate at which a single chemical species combines to produce a new species, such as methyl radicals combining in a gas to form ethane molecules. See Atkins [**?**, p. 134].
$\frac{d^2y}{dx^2} = \frac{C}{L}\sqrt{\left(\frac{AC}{L}\right)^2 + \left(\frac{dy}{dx}\right)^2}$	The graph of the solution models the shape of the *Gateway Arch* in St. Louis, where y is its height at a distance x from one end of its base. The constants A, C, and L relate the lengths of the base, top, and centroid. The Gateway Arch has the shape of an inverted *catenary*. A ***catenary*** is a curve that has the shape of a chain suspended from two points at the same level. The equation used to design the catenary curve shape of Arch can be found at the website: *www.nps.gov/jeff*.

Differential Equation	**Situation**
$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F(t)$	This equation models the motion of a damped mass-spring system subjected to a time-dependent force $F(t)$.
$\frac{\hbar^2}{2m}\cdot\frac{d^2\psi}{dx^2}+\left(E-\frac{1}{2}kx^2\right)\psi=0$	This equation from quantum mechanics is the time-independent *Schrödinger's equation* for the one-dimensional simple harmonic oscillator. The constant $\hbar$ is defined in terms of Planck's constant h by $\hbar=h/2\pi$.
$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\lambda(\lambda+1)y=0$	This equation is known as *Legendre's differential equation* and is one of several equations used for calculating the energy levels of the hydrogen atom.
$EI\frac{d^4y}{dx^4}=w(x)$	This differential equation models the vertical displacement $y(x)$ of a point located a distance x from the fixed end of a beam of uniform cross section, where $w(x)$ represents the load at x; E and I are constants.
$x''-\varepsilon(1-x^2)x'+x=0$	The *van der Pol equation* models the current at time t in an electrical circuit with nonlinear resistance.
$4xy^2\left(y^{(4)}\right)^3-3x^4y^5\left(y''\right)^6=\cos^9(x^{10})$	This is just one mean-looking equation concocted by the author.
$\frac{dx}{dt}=\sigma(y-x)$ $\frac{dy}{dt}=rx-y-xz$ $\frac{dz}{dt}=xy-bz$	This set of three differential equations, called the *Lorenz system*, is a overly simplified version of a complicated system of twelve equations used to model convection in the atmosphere. The Lorenz system models the chaotic rotational motion of a wheel with leaking compartments of water symmetrically positioned around its rim. See Appendix A for more information.

Even though the equations in Table 1.1 come from diverse fields, they do have some common features. The foremost feature shared by all of them is that they have at least one derivative, which is precisely what makes them differential equations in the first place! We make special note of this by formally defining what is meant by a differential equation.

Differential Equation

Definition 1.1. A ***differential equation*** is an equation that involves one or more derivatives of some unknown function or functions.

To complicate matters, there are various types of differential equations: chief among them are *ordinary differential equations*, *partial differential equations*, and *integro-differential equations*. The equations in Table 1.1 are all examples of *ordinary differential equations* because they only involve *ordinary derivatives*. Ordinary derivatives are the derivatives that we study in a first (single-variable) calculus course. *Partial differential equations* are equations involving derivatives called *partial derivatives*—how a partial derivative differs from an ordinary derivative is discussed later on, after the review of ordinary derivatives in the next section. To give us an inkling of what partial differential equations look like, here is a classic example:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

It is used to model the conduction of heat through an extremely thin metal bar, where $u(x, t)$ is the temperature at the point x in the bar at time t.

Integro-differential equations involve not only derivatives of unknown functions but also their integrals. For example, in Chapter **??** we will solve integro-differential equations that look like

$$x'(t) = f(t) + \int_0^t k(t-u)x(u)\,du.$$

This book is devoted to a study of ordinary differential equations. Even so, there will be brief forays at times into topics involving very simple partial differential equations, integral equations, and integro-differential equations.

Before we formally define what is meant by an ordinary differential equation, let's point out some features that the equations in Table 1.1 have in common. First we observe that each equation in Table 1.1 contains a single ***independent variable*** and one or more ***dependent variables***. It is a relatively simple matter to tell these two types of variables apart from the derivatives themselves, since differentiation always takes place with respect to the independent variable. Obviously then, the other variable, the one being differentiated, is the dependent variable.

Example 1.1. The first entry in Table 1.1 is the Malthusian law of population growth:

$$\frac{dp}{dt} = rp.$$

Translated into words, the equation says that the rate at which the current population p of an organism changes with respect to the time t is equal to the product of the constant r and the current population p. The time t is the independent variable and the population p is the dependent variable.

Example 1.2. The equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

in Table 1.1 models the vertical displacement of a beam. Since y is differentiated with respect to x, the independent variable is x and the dependent one is y.

The name *dependent variable* is befitting because it describes the type of variable it is: it depends in some functional way on the independent variable as prescribed by the differential equation, although this dependence is not always possible. For example,

$$y^2 + (y')^2 = -1$$

is a differential equation; even so, no real-valued function[2] can fulfill the prescript that the sum of its square and the square of its derivative is equal to a negative number.

Space on a page can be saved by replacing ***Leibniz notation***, which uses the Latin "d" for denoting derivatives, such as

$$\frac{dp}{dt}, \quad \frac{d^2x}{dt^2}, \quad \frac{d^4y}{dx^4},$$

with a shorthand notation that uses primes ($'$) or overdots ($\dot{}$) for differentiation. In ***prime notation***, the derivatives

$$\frac{dy}{dx} \quad \text{and} \quad \frac{dp}{dt}$$

are written

$$y' \quad \text{and} \quad p',$$

respectively. A shortcoming of this notation is that the independent variable is not explicitly stated.

The ***overdot notation*** is reserved for derivatives that are taken with respect to the time t. For example, $\dot{p}$ means dp/dt. Thus, the Malthusian population law

$$\frac{dp}{dt} = rp$$

in the overdot notation becomes

$$\dot{p} = rp.$$

We also have to be aware of the *orders* of the derivatives appearing in equations. The derivatives

$$\frac{dy}{dx}, \quad \dot{p}, \quad z'$$

are ***first-order derivatives***, whereas the derivatives

$$\frac{d^2x}{dt^2}, \quad y'', \quad \ddot{p}$$

[2] A function is ***real-valued*** when every evaluation of it results in a real number. Even though the function $i \sin x$, where $i^2 = -1$, satisfies the differential equation, it is a complex-valued solution, not a real-valued solution.

are ***second-order derivatives***. Of course, some differential equations have derivatives of even higher order: ***third-order derivatives*** such as

$$y''', \quad \frac{d^3x}{dt^3}$$

or ***fourth-order derivatives*** such as

$$\frac{d^4x}{dt^4}$$

or even higher. It is easy to lose track of the number of primes or overdots when the order is more than three. In such a case, it is customary to use either Leibniz notation or to use superscripts enclosed in parentheses to denote such derivatives: for example, d^4y/dx^4 or $y^{(4)}$ is preferred over y''''. The nth derivative of y with respect to x is written as d^ny/dx^n or as $y^{(n)}$.

All of the previous derivatives are known as *ordinary derivatives*. When we take the ordinary derivative of a function, the term *ordinary* indicates that we are dealing with a function of a single variable. In other words, an ***ordinary derivative*** is a derivative of a function of a single independent variable with respect to that variable. The word *ordinary* qualifies the word *derivative*, distinguishing between the derivatives of single-variable calculus from the ones of multivariable calculus. Multivariable calculus deals with functions of two or more variables; their derivatives are called *partial derivatives*. They will be introduced shortly; but for now, let's review the definition and meaning of the ordinary derivative of a function.

1.1.1 Ordinary Derivatives

Let's review the meaning of an *ordinary derivative* with an example. Imagine stretching a filament-like copper wire of length 25 centimeters tautly along a straight line. Let the line serve as the x-axis and the left end of the wire designate the location of the origin. Suppose that the wire is heated unevenly in such a way that each of its points eventually reaches a constant temperature but that the temperature generally varies from point to point. Even though in reality the wire is a three-dimensional object, its very thinness suggests that variations in temperature along the y- and z-directions are negligible. Consequently, the wire may be regarded ideally as a one-dimensional mathematical object: the line segment extending from $x = 0$ cm to $x = 25$ cm. Now suppose that Table 1.2 gives temperature measurements, accurately to the ten-thousandth place, at 5-centimeter intervals along the wire.

Table 1.2: **Temperatures at Points of an Unevenly Heated Wire**

***x* (cm)**	0	5	10	15	20	25
T (°C)	100.0000	99.8740	99.4980	98.8720	97.9960	96.8700

The ***average rate of change*** of the temperature with respect to x, as x changes from 10 cm to 15 cm, is given by the *difference quotient* $\Delta T/\Delta x$, where ΔT is the change in the temperature corresponding to Δx, the change in the x-coordinate. Thus,