Math 319 HW3 Solution to selected problems

Sec 2.3 #10 A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
(a) Determine the maximum amount that this buyer can afford to borrow.
(b) Determine the total interest paid during the term of the mortgage.

Solution:
(a) We let $N(t)$ to denote the money you owe the bank at time t. So $N(0)$ is the original loan amount that we are looking for. Here the term of the mortgage is 20 years, which means that we hope to exactly pay off all the loan at $t = 20$. In other words, we have

$$N(20) = 0.$$

Now let us find the differential equation satisfied by N. Note that the mortgage has a 9% interest rate, and it is compounded continuously. This means if you are not paying any money on your mortgage, then the money you owe the bank will simply grows exponentially, i.e.

$$N'(t) = 0.09N.$$

However, unfortunately we have to make monthly payments on the mortgage. In the question, it says that the monthly payment is \$800/month, and it is made on a continuous basis too. This means every year we are paying \$9600/year continuously for the mortgage, and "paying the mortgage" decreases the money we owe the bank. That's why we subtract 9600 on the right hand side of the ODE, and it becomes

$$N'(t) = 0.09N - 9600.$$

Finally, we get an "initial value problem"

$$\begin{cases} N'(t) = 0.09N - 9600 \\ N(20) = 0. \end{cases} \tag{1}$$

Note that here the "initial condition" is given at $t = 20$ instead of $t = 0$, and indeed we will use it to find out $N(0)$. Let us try to solve this ODE. It is a first order linear equation, and it is also separable, so you could use either way to solve it. I'll use the method for separable equations here, but it's perfectly fine if you prefer the other way.

First write it as

$$\frac{dN}{0.09N - 9600} = dt,$$

then integrate both sides:

$$\frac{1}{0.09} \ln |0.09N(t) - 9600| = t + C$$

So finally we have the general solution is

$$N(t) = Ce^{0.09t} + \frac{9600}{0.09},$$

here C is an arbitrary constant. In order to satisfy the initial condition $N(20) = 0$, we plug in $t = 20$, and get

$$N(20) = Ce^{0.09\times20} + \frac{9600}{0.09} = 0,$$

solve for C, and we get $C \approx -17632$. This means our solution for this IVP is

$$N(t) = -17632e^{0.09t} + \frac{9600}{0.09},$$

and this means our original loan amount, i.e. $N(0)$, is given by

$$N(0) = -17632e^{0.09\times0} + \frac{9600}{0.09} \approx 89035.$$

(b) Now we want to figure out the total interest we have paid to the bank. This is easy once we get the original loan amount in part (a). This is because the total interest is simply

total interest you paid the bank = total money you paid the bank − amount of the original loan

Due to (a), the original loan amount is \$89035. Also, since we are making a \$800 monthly payment for 20 years, the total money we paid the bank is

$$\$800 \times 12 \times 20 = \$192,000.$$

Therefore,

$$\text{total interest you paid the bank} = \$192,000 - \$89,035 = \$102,965.$$

You can see that the total interest is even higher than your original loan amount! (Therefore one should never apply for a mortgage with interest rate that high...)

Sec 2.3 #13 The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

Solution:

Let us denote by $N(t)$ the population of mosquitos at the t-th day. The question mentions that in the absence of predators, $N(t)$ would grow at a rate proportional to itself, and $N(t)$ doubles each week. Let's try to interpret what that means.

"$N(t)$ grow at a rate proportional to itself" means without predators, $N(t)$ would satisfy the equation

$$N'(t) = rN(t),$$

where r is the growth rate constant that we do not know yet. We will use the next information to figure it out:

"in the absence of predators, N(t) doubles each week". This means for the equation above, (where we does not take any predator into account), we should have

$$N(7) = 2N(0),$$

meaning that the population at day 7 should be two times the initial population.

We will use this to solve for r. Note that the equation $N'(t) = rN(t)$ leads to the general solution $N(t) = Ce^{rt}$, hence we have

$$N(0) = C, \quad N(7) = Ce^{7r},$$

and $N(7) = 2N(0)$ leads to

$$Ce^{7r} = 2C.$$

Now you can see that even we don't know what is C, it does not matter since we can cancel C from both sides and solve for r, and finally we get $7r = \ln 2$, meaning that

$$r = \frac{\ln 2}{7}.$$

Plug it back to the differential equation, and we will get $N(t)$ satisfies

$$N'(t) = \frac{\ln 2}{7} N(t)$$

in the absence of all the predators.

Now let's bring in the birds (or rats, or pest control?). They will eat 20,000 mosquitos a day, which means that in the reality $\frac{dN}{dt}$ should consist of two parts, one is the right hand side we used to have, and the other one is the rate at which mosquitos get eaten. Therefore the ODE should be modified into

$$N'(t) = \frac{\ln 2}{7} N(t) - 20000,$$

and I will leave it to you to solve this first order linear equation. Once you solve for $N(t)$, that would be the population of the mosquitos at time t.