Differential Equation for Loan Repayment

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Abstract

In some ways this is a set of notes from our explorations the first week of PH213 this year. This document will mostly seek to highlight the results we found so that we have a common starting point for the homework questions.

Introduction

Differential equations are important because there are so many aspects of physics and engineering that are easily described by differential equations. Not the least of these is Newtons 2nd Law. Here we explore the mathematics of loans (mortgages) as a moderately understandable example of applied differential equations.

The Equation

Here's the basic description of a loan with that we might be repaying.

$$\frac{dP(t)}{dt} = r \cdot P(t) - M \tag{1}$$

where $P(t)$ is the current value of the principal of the loan, r is the interest rate for the compounding period, and M is the payment that we would make during the same compounding period. Is is a useful exercise to go though and make sure that the units of each term are the same. You will notice that they are all in terms of $\frac{dollars}{time}$ where in most cases we think of repayment as a monthly activity where time is in months.

This can lead to lots of dimensional confusion since we usually are given an interest rate on the loan based on an annual rate while the interest is most typically (though not always) calculated on a monthly basis..

Everything gets a good bit more complex if we worry about adjustable rate loans where r is not constant but we will sidestep this concern at this point. This equation is usually rewritten in math courses as

$$\frac{dP(t)}{dt} - r \cdot P(t) = M \tag{2}$$

so that we can separate out the homogeneous form of the equation

$$\frac{dP(t)}{dt} - r \cdot P(t) = 0 \tag{3}$$

The Solution

The usual approach to finding a function which is the solution to equation 2 is to first find a solution to equation 3. After we have such a solution we will then seek to cleverly patch it up so it also works for the actual inhomogeneous equation. In physics and engineering we often start with what we call a "trial solution". This is something that we think (hope) has the right general characteristics to be a solution. Because equation 3 essentially says that the derivative of the function $P(t)$ and the original function are the same (their difference is 0!) the exponential function is a reasonable choice. The resumed variable, because of the derivative, is t in units of months.

$$P(t) = A \cdot e^{kt} + C \tag{4}$$

In this trial solution A and C are just constants which we put in as "adjustment knobs" that allows us to "tune up" our solution to, possibly, make it work. If you take the derivative of this trial solution and the substitute everything back into the homogeneous form [eqn 3] of the equation then you find, after some cancellation that

$$k = r \quad \text{and} \quad C = 0 \tag{5}$$

are required to satisfy the equation. This tells us that k is the interest rate.

If we now take our trial solution, k = r, and put it into the inhomogeneous equation [eqn 2] you find that

$$-r \cdot C = -M \quad \text{or} \quad C = M/r \tag{6}$$

This leads to the following general solution to the differential equation which is true for all loans with fixed interest rates and fixed payments made for each compounding interval.

Initial/Boundary Conditions

You might feel like this isn't very useful since there are many different settings in which we take out loans for very different amounts of money. In the language of differential equations these are just different boundary conditions that must be satisfied by the solution. Initial conditions are a special case of a boundary condition that is based on the characteristic of the solution at $t = 0$.

For a loan $P(t = 0)$ is the amount of money we borrow. We might well call it P_0 in the same way that x_0 is the initial position of an object. If we evaluate our solution at $t = 0$ and are careful about the math we find that

$$\begin{aligned} P_0 &= A \cdot e^{r \cdot 0} + \tfrac{M}{r} \\ &\text{or} \\ A &= P_0 - \tfrac{M}{r} \end{aligned} \tag{7}$$

If we know P_0 and r there are still 2 unknowns in this equation. Our other constraint might well be that we know the life of the loan. In this case $P(t = T) = 0$, where T is the life of the loan in months, is another constraint. When we apply this constraint we find

$$\begin{aligned} 0 &= A \cdot e^{r \cdot T} + \tfrac{M}{r} \\ &\text{or} \\ A &= -\tfrac{M}{r} \cdot e^{-r \cdot T} \end{aligned} \tag{8}$$

Substituting equation 8 into equation 7 and solving we get the following useful relationships.

$$\begin{aligned} P(t) &= \tfrac{M}{r}(1 - e^{-r(T-t)}) \\ &\text{where} \\ M &= \tfrac{P_0 \cdot r}{1 - e^{-rT}} \end{aligned} \tag{9}$$

Final Thoughts

With this information you should be able to work out a whole range of possible loan scenarios if you are careful and understand which terms you "know" and which terms you are calculating at any point. Remember that r is actual the annual interest rate/12 if you are doing monthly payment and compounding which is the normal approach.