Justification for the Amount of an Annuity

In your project, you encountered the formula for the amount of an annuity after ***x*** years, into which ***P*** dollars is placed every year, and which earns and interest rate of ***r***:

$$\frac{P(1+r)}{r}\left((1+r)^x - 1\right)$$

This assumes that interest is only paid once per year, right before you make the next year's deposit. Here's why the formula works: It's easier to see why this formula works if we build the value of the annuity up from the beginning. Here's a two column chart to show how it starts:

Year	Amount of Annuity
0	P
1	$P + rP + P = 2P + rP$
2	$2P + rP + r(2P + rP) + P = 3P + rP + r(2P + rP)$

Note that the ***P*** represents the deposit that you make at the end of each year. Now, looking at each of those quantities exponentially, we find that they are equal to the following:

Year	Amount of Annuity
0	P
1	$P + rP + P = 2P + rP = P + P(1+r)$
2	$2P + rP + r(2P + rP) + P = 3P + rP + r(2P + rP)$
	$= P + P(1+r) + P(1+r)^2$

A-ha! Let's only look at those new entries:

Year	Amount of Annuity
0	P
1	$P + P(1+r)$
2	$P + P(1+r) + P(1+r)^2$

Now, if you ponder this for a moment, it should make perfect sense: each year, the same interest rate ***r*** is being applied to whatever money is in the account; however, some money in the account has been there longer than other money. Therefore, the sum of the terms shows that some money has been compounded more than others. Continuing:

Year	Amount of Annuity
0	P
1	$P + P(1+r)$
2	$P + P(1+r) + P(1+r)^2$
3	$P + P(1+r) + P(1+r)^2 + P(1+r)^3$
4	$P + P(1+r) + P(1+r)^2 + P(1+r)^3 + P(1+r)^4$
…	…
t	$P + P(1+r) + P(1+r)^2 + \ldots + P(1+r)^x$

Now, as cool as this is, it's a pretty messy formula, as you'd have to individually add up all of the terms to get the value. Here's where some cool algebra comes in. Let the value of the annuity at t years be called ***A***. Thus,

$$A = P + P(1+r) + P(1+r)^2 \ldots + P(1+r)^x$$

Call that equation 2. To get equation 1, multiply equation 2 by (1+***r***):

$$\begin{aligned}(1+r)A &= (1+r)(P + P(1+r) + P(1+r)^2 \ldots + P(1+r)^x) \\ (1+r)A &= P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots + P(1+r)^{x+1}\end{aligned}$$

Still with me? Good! Now, subtract equation 2 from equation 1. You'll get what's called a "telescoping" series, in that many, many of the terms cancel:

$$\begin{array}{r}(1+r)A = P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots + P(1+r)^{x+1} \\ -\left(A = P + P(1+r) + P(1+r)^2 \ldots + P(1+r)^x\right) \\ \hline\end{array}$$

$$\begin{aligned}(1+r)A - A &= \cancel{P(1+r)} + \cancel{P(1+r)^2} + \cancel{P(1+r)^3} + \ldots + P(1+r)^{t+1} - \left(P + \cancel{P(1+r)} + \cancel{P(1+r)^2} \ldots + \cancel{P(1+r)^x}\right) \\ A(1+r-1) &= P(1+r)^{x+1} - P \\ Ar &= P\left((1+r)^{x+1} - 1\right) \\ A &= \frac{P}{r}\left((1+r)^{x+1} - 1\right)\end{aligned}$$

And would you just look at that? Pretty cool, eh? We need, however, to make one final adjustment: at the end of the annuity, you won't make the final deposit of ***P*** dollars, since you'll be withdrawing the money. So,

$$\begin{aligned}A &= \frac{P}{r}\left((1+r)^{x+1} - 1\right) - P \\ &= P\left(\frac{1}{r}\left((1+r)^{x+1} - 1\right) - 1\right) \\ &= P\left(\frac{1}{r}\left((1+r)^{x+1} - (1+r)\right)\right) \\ &= \frac{P(1+r)}{r}\left((1+r)^x - 1\right)\end{aligned}$$

And there you have it!