A Justification for the Fundamental Counting Principle (FCP)

The FCP is a very nice way of enumerating large, cumbersome event or sample spaces by making the use of a multiplicative shortcut:

> **The Fundamental Counting Principle**
>
> Suppose Event A can happen in one of ***x*** ways, followed by Event B that can happen in one of ***y*** ways. The number of ways that we can do both events jointly (that is, Event A followed by Event B) is given by $\boldsymbol{x*y}$.
>
> This relationship holds for more than two events, as well.

This paper will prove this result[1].

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Let's prove it for the 2 – event case first. Event 1 can happen in one of ***x*** ways, so let's name each of these possibilities: A_1, A_2, A_3,…,A_{x-1}, A_x. Also, Event B can happen in one of ***y*** ways, so we can name each of the ways that Event B can happen as B_1, B_2, B_3, …,$B_{y-1,}$ B_y.

One way to construct all possible joint events would be to exhaustively list them. I'll start by listing all possible ways of combining A_1 with the elements in Event B, then move to A_2, A_3, and so on, until finished. While tedious, it gets the job done (ellipses in either direction, up or down, means a pattern is continued):

A_1B_1	A_1B_2	A_1B_3	…	A_1B_{y-1}	A_1B_y
A_2B_1	A_2B_2	A_2B_3	…	A_2B_{y-1}	A_2B_y
A_3B_1	A_3B_2	A_3B_3	…	A_3B_{y-1}	A_3B_y
			…		
$A_{x-1}B_1$	$A_{x-1}B_2$	$A_{x-1}B_3$	…	$A_{x-1}B_{y-1}$	$A_{x-1}B_y$
A_xB_1	A_xB_2	A_xB_3	…	A_xB_{y-1}	A_xB_y

Now, let's ask ourselves: how many rows are there in this grid of outcomes? Well, since there are ***x*** possibilities for each element of Event A, there must be ***x*** rows…one for each element of Event A. Similarly, since there are ***y*** elements in Event B's possibilities, there must be ***y*** columns in this grid:

y columns

x rows

A_1B_1	A_1B_2	A_1B_3	…	A_1B_{y-1}	A_1B_y
A_2B_1	A_2B_2	A_2B_3	…	A_2B_{y-1}	A_2B_y
A_3B_1	A_3B_2	A_3B_3	…	A_3B_{y-1}	A_3B_y
			…		
$A_{x-1}B_1$	$A_{x-1}B_2$	$A_{x-1}B_3$	…	$A_{x-1}B_{y-1}$	$A_{x-1}B_y$
A_xB_1	A_xB_2	A_xB_3	…	A_xB_{y-1}	A_xB_y

To find the total number of cells in the grid (which would be the set of all joint events), we can just find the area of the grid, using the old "length times width" convention:

Area of grid = length of grid time width of grid = $\boldsymbol{x*y}$.

[1] Being a math geek, I like to prove whatever I can.

And there you have it. An alternate way (for those not geometrically welcoming) would be to use an extension/bastardization of the distributive property:

A_1B_1	A_1B_2	A_1B_3	…	A_1B_{y-1}	A_1B_y
A_2B_1	A_2B_2	A_2B_3	…	A_2B_{y-1}	A_2B_y
A_3B_1	A_3B_2	A_3B_3	…	A_3B_{y-1}	A_3B_y
			…		
$A_{x-1}B_1$	$A_{x-1}B_2$	$A_{x-1}B_3$	…	$A_{x-1}B_{y-1}$	$A_{x-1}B_y$
A_xB_1	A_xB_2	A_xB_3	…	A_xB_{y-1}	A_xB_y

$$\left.\begin{array}{c} A_1 (B_1, B_2, B_3, \ldots B_{y-1}, B_y) \\ A_2 (B_1, B_2, B_3, \ldots B_{y-1}, B_y) \\ A_3 (B_1, B_2, B_3, \ldots B_{y-1}, B_y) \\ \ldots \\ A_{x-1} (B_1, B_2, B_3, \ldots B_{y-1}, B_y) \\ A_x (B_1, B_2, B_3, \ldots B_{y-1}, B_y) \end{array}\right\} \quad \left.\begin{array}{c} A_1 (\boldsymbol{y} \text{ terms}) \\ A_2 (\boldsymbol{y} \text{ terms}) \\ A_3 (\boldsymbol{y} \text{ terms}) \\ \ldots \\ A_{x-1} (\boldsymbol{y} \text{ terms}) \\ A_x (\boldsymbol{y} \text{ terms}) \end{array}\right\} \quad (A_1, A_2, A_3, \ldots A_{x-1}, A_x) * (\boldsymbol{y} \text{ terms})$$

$$\downarrow$$

$$(\boldsymbol{x} \text{ terms}) * (\boldsymbol{y} \text{ terms})$$

$$\downarrow$$

$$\boldsymbol{x * y}$$

While this method is algebraically cumbersome, it does allow us to move upwards to as many different events as we wish (the geometric illustration, while neat, breaks down visually after 3 events, since we live in a 3 – dimensional world).