Justification for the mean, variance and standard deviation of the binomial random variable

We have learned that, for the binomial random variable, $P(x) = \binom{n}{x} p^x q^{n-x}$,

$$\mu = np \qquad \sigma^2 = npq \qquad \sigma = \sqrt{npq}$$

where $\boldsymbol{p}$ is the probability of success, $\boldsymbol{q}$ is the probability of failure, and $\boldsymbol{n}$ is the number of trials. The purpose of this paper is to justify these formulas so you don't think they just fell out of the sky. We need two tools to help us with this. One is the simple fact that $p + q = 1$. That shouldn't be too hard to believe; by definition, $\boldsymbol{p}$ and $\boldsymbol{q}$ are complementary. The other tool is called the binomial theorem, which is stated mathematically as follows, for real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ and positive integers $\boldsymbol{n}$:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

First, we'll justify that $\mu = np$. We need to use the formula for the binomial listed above, $P(x) = \binom{n}{x} p^x q^{n-x}$, and also the definition of the mean of a random variable:

$$\mu = \sum x \cdot P(x) = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + \ldots + x_n \cdot P(x_n)$$

Combining these two ideas, and doing some ***heavy*** duty algebra,

$$\mu = 0\binom{n}{0} p^0 q^n + 1\binom{n}{1} p^1 q^{n-1} + 2\binom{n}{2} p^2 q^{n-2} + 3\binom{n}{3} p^3 q^{n-3} + \ldots + (n-1)\binom{n}{n-1} p^{n-1} q^1 + n\binom{n}{n} p^n q^0$$

Using the definition of combination,

$$\mu = npq^{n-1} + 2\frac{n!}{(n-2)!2!} p^2 q^{n-2} + 3\frac{n!}{(n-3)!3!} p^3 q^{n-3} + \ldots + (n-1)\frac{n!}{(n-1)!} p^{n-1} q^1 + np^n$$

$$\mu = npq^{n-1} + \frac{n!}{(n-2)!} p^2 q^{n-2} + \frac{n!}{(n-3)!2!} p^3 q^{n-3} + \ldots + \frac{n!}{(n-2)!} p^{n-1} q^1 + np^n$$

Factoring out ***np***,

$$\mu = np\left[q^{n-1} + \frac{(n-1)!}{(n-2)!}pq^{n-2} + \frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + \ldots + \frac{(n-1)!}{(n-2)!}p^{n-2}q + p^{n-1}\right]$$

Using the binomial theorem, the bracketed terms reduce to $(p+q)^{n-1}$. So, therefore,

$$\mu = np(p+q)^{n-1} = np \qquad \text{Q.E.D.}$$

Next, we'll justify that $\sigma^2 = npq$. We need, again, the binomial formula listed above as well as the definition of the variance of a random variable:

$$\sigma^2 = \sum(x-\mu)^2 \cdot P(x) = (x_1-\mu)^2 \cdot P(x_1) + (x_2-\mu)^2 \cdot P(x_2) + \ldots + (x_n-\mu)^2 \cdot P(x_n)$$

Now, stay with me…this gets a little crazy!

$$\sigma^2 = (0-\mu)^2 \cdot \binom{n}{0}p^0q^n + (1-\mu)^2 \cdot \binom{n}{1}p^1q^{n-1} + (2-\mu)^2 \cdot \binom{n}{2}p^2q^{n-2} + \ldots + ((n-1)-\mu)^2 \cdot \binom{n}{n-1}p^{n-1}q + (n-\mu)^2 \cdot \binom{n}{n}p^n$$

$$\sigma^2 = (0-np)^2 \cdot \binom{n}{0}p^0q^n + (1-np)^2 \cdot \binom{n}{1}p^1q^{n-1} + (2-np)^2 \cdot \binom{n}{2}p^2q^{n-2} + \ldots + ((n-1)-np)^2 \cdot \binom{n}{n-1}p^{n-1}q + (n-np)^2 \cdot \binom{n}{n}p^n \quad \text{(since } \pi = np\text{)}$$

Each of the $(x-np)^2$ factors can be expanded (using the distributive property) into $x^2 - 2xnp + n^2p^2$. So we now have:

$$\sigma^2 = n^2p^2q^n + (1-2np+n^2p^2) \cdot \binom{n}{1}p^1q^{n-1} + (4-4np+n^2p^2) \cdot \binom{n}{2}p^2q^{n-2} + \ldots + ((n-1)^2 - 2(n-1)np + n^2p^2) \cdot \binom{n}{n-1}p^{n-1}q + (n^2 - 2n \cdot np + n^2p^2) \cdot \binom{n}{n}p^n$$

Now, by distributing each trinomial to its corresponding $P(x)$ factor, collecting like terms, and factoring, we have

$$\sigma^2 = \left[\binom{n}{1}p^1q^{n-1} + 4\binom{n}{2}p^2q^{n-2} + \ldots + (n-1)^2\binom{n}{n-1}p^{n-1}q + n^2p^n\right] \qquad \text{(A)}$$

$$-2np\left[\binom{n}{1}p^1q^{n-1} + 2\binom{n}{2}p^2q^{n-2} + \ldots + (n-1)\binom{n}{n-1}p^{n-1}q + np^n\right] \qquad \text{(B)}$$

$$+n^2p^2\left[q^n + \binom{n}{1}p^1q^{n-1} + \ldots + \binom{n}{n-1}p^{n-1}q^1 + p^n\right] \qquad \text{(C)}$$

We will deal with each of A, B, and C in turn to facilitate the totally awesome algebra. Start with C:

$$\text{C} = n^2p^2\left(q^n + \binom{n}{1}p^1q^{n-1} + \ldots + \binom{n}{n-1}p^{n-1}q^1 + p^n\right)$$

Using the binomial theorem, the terms in parentheses are simply the expansion of $(p+q)^n$. Therefore:

$$\text{C} = n^2p^2\left(q^n + \binom{n}{1}p^1q^{n-1} + \ldots + \binom{n}{n-1}p^{n-1}q^1 + p^n\right) = n^2p^2(p+q)^n = n^2p^2$$

Next, let's deal with B:

$$\text{B} = -2np\left[\binom{n}{1}p^1q^{n-1} + 2\binom{n}{2}p^2q^{n-2} + \ldots + (n-1)\binom{n}{n-1}p^{n-1}q + np^n\right]$$

Using the definition of a combination, we have

$$\text{B} = -2np\left[\frac{n!}{(n-1)!}p^1q^{n-1} + 2\frac{n!}{(n-2)!2!}p^2q^{n-2} + \ldots + (n-1)\frac{n!}{(n-1)!}p^{n-1}q + np^n\right]$$

$$\text{B} = -2np\left[\frac{n!}{(n-1)!}p^1q^{n-1} + \frac{n!}{(n-2)!}p^2q^{n-2} + \ldots + \frac{n!}{(n-2)!}p^{n-1}q + np^n\right]$$

Now, factoring out ***np***,

$$\text{B} = -2n^2p^2\left[q^{n-1} + \frac{(n-1)!}{(n-2)!}p^1q^{n-2} + \ldots + \frac{(n-1)!}{(n-2)!}p^{n-2}q + p^{n-1}\right]$$

and what we have left in the brackets is simply $(p+q)^{n-1}$ by the binomial theorem. So,

$$\text{B} = -2n^2p^2\left[q^{n-1} + \frac{(n-1)!}{(n-2)!}p^1q^{n-2} + \ldots + \frac{(n-1)!}{(n-2)!}p^{n-2}q + p^{n-1}\right] = -2n^2p^2(p+q)^{n-1} = -2n^2p^2$$

Finally, let's work on A:

$$\text{A} = \binom{n}{1}p^1q^{n-1} + 4\binom{n}{2}p^2q^{n-2} + 9\binom{n}{3}p^3q^{n-3} + \ldots + (n-1)^2\binom{n}{n-1}p^{n-1}q + n^2p^n$$

Using the definition of combination,

$$\text{A} = \frac{n!}{(n-1)!}p^1q^{n-1} + 4\frac{n!}{(n-2)!2!}p^2q^{n-2} + 9\frac{n!}{(n-3)!3!}p^3q^{n-3} \ldots + (n-1)^2\frac{n!}{(n-1)!}p^{n-1}q + n^2p^n$$

Now comes a subtle step…make sure you follow it!

$$\text{A} = \frac{n!}{(n-1)!}p^1q^{n-1} + 2^2\frac{n!}{(n-2)!2!}p^2q^{n-2} + 3^2\frac{n!}{(n-3)!3!}p^3q^{n-3} + \ldots + (n-1)^2\frac{n!}{(n-1)!}p^{n-1}q + n^2p^n$$

$$\text{A} = \frac{n!}{(n-1)!}p^1q^{n-1} + 2\frac{n!}{(n-2)!}p^2q^{n-2} + 3\frac{n!}{(n-3)!2!}p^2q^{n-2} + \ldots + (n-1)\frac{n!}{(n-2)!}p^{n-1}q + n^2p^n$$

Next, factor out a common ***np***:

$$A = np\left[\frac{(n-1)!}{(n-1)!}q^{n-1} + 2\frac{(n-1)!}{(n-2)!}p^1q^{n-2} + 3\frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + \ldots + (n-1)\frac{(n-1)!}{(n-2)!}p^{n-2}q + np^{n-1}\right]$$

Now, again comes a subtle step:

$$A = np\left[\frac{(n-1)!}{(n-1)!}q^{n-1} + \left(\frac{(n-1)!}{(n-2)!}p^1q^{n-2} + \frac{(n-1)!}{(n-2)!}p^1q^{n-2}\right) + \left(\frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + 2\frac{(n-1)!}{(n-3)!2!}p^2q^{n-3}\right) + \ldots\right.$$
$$\left.\ldots + \left(\frac{(n-1)!}{(n-2)!}p^{n-2}q + (n-2)\frac{(n-1)!}{(n-2)!}p^{n-2}q\right) + \left(p^{n-1} + (n-1)p^{n-1}\right)\right]$$

This seems insane, but upon rewriting, I'll have a nice binomial collapse:

$$A = np\left[\left[q^{n-1} + \frac{(n-1)!}{(n-2)!}p^1q^{n-2} + \frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + \ldots + \frac{(n-1)!}{(n-2)!}p^{n-2}q + p^{n-1}\right] + \ldots\right.$$
$$\left.\ldots + \left[\frac{(n-1)!}{(n-2)!}p^1q^{n-2} + 2\frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + \ldots + (n-2)\frac{(n-1)!}{(n-2)!}p^{n-2}q + (n-1)p^{n-1}\right]\right]$$

Notice that the first "bracketed within the brackets" grouping is simply the binomial expansion of $(p+q)^{n-1}$:

$$A = np\left[1 + \frac{(n-1)!}{(n-2)!}p^1q^{n-2} + 2\frac{(n-1)!}{(n-3)!2!}p^2q^{n-3} + \ldots + (n-2)\frac{(n-1)!}{(n-2)!}p^{n-2}q + (n-1)p^{n-1}\right]$$

What's that? You want one more subtle step? OK! Factor out $(n-1)p$ from the inner bracketed series:

$$A = np\left[1 + (n-1)p\left[\frac{(n-2)!}{(n-2)!}q^{n-2} + 2\frac{(n-2)!}{(n-3)!2!}pq^{n-3} + \ldots + (n-2)\frac{(n-2)!}{(n-2)!}p^{n-3}q + p^{n-2}\right]\right]$$

$$A = np\left[1 + (n-1)p\left[q^{n-2} + \frac{(n-2)!}{(n-3)!}pq^{n-3} + \ldots + \frac{(n-2)!}{(n-3)!}p^{n-3}q + p^{n-2}\right]\right]$$

And, once again, the terms in the innermost brackets are simply the binomial expansion of $(p+q)^{n-2}$. So we now have:

$$\mathrm{A} = np\left[1+(n-1)p\left[q^{n-2}+\frac{(n-2)!}{(n-3)!}pq^{n-3}+\ldots+\frac{(n-2)!}{(n-3)!}p^{n-3}q+p^{n-2}\right]\right] = np\left[1+(n-1)p(p+q)^{n-2}\right] = np\left[1+(n-1)p\right]$$

Now the fun part! Remember that $\sigma^2 = \mathrm{A}+\mathrm{B}+\mathrm{C}$, so:

$$\sigma^2 = np\left[1+(n-1)p\right]-2n^2p^2+n^2p^2$$

Reducing,

$$\sigma^2 = np\left[1+np-p\right]-n^2p^2 = np+n^2p^2-np^2-n^2p^2 = np-np^2 = np(1-p)$$

Recall, however, that $p+q=1$. So, we now have that

$$\sigma^2 = np(1-p) = npq \qquad \text{Q.E.D.}$$

It follows immediately that the standard deviation, σ, is equal to the square root of the variance, or $\sqrt{npq}$.