A Justification for the Poisson Probability Function

In practice, the Poisson probability function can sometimes be very handy. You might remember from your distribution project that it models many things. Its function

$$p(\mu, x) = \frac{e^{-\mu}\mu^x}{x!}$$

can be derived (with a little algebra and calculus) from the binomial function we studied in class:

$$b(n, p, x) = \binom{n}{x} p^x q^{n-x}$$

The Poisson is actually a consequence of the binomial. That is, under certain conditions, the binomial will become Poisson. The conditions? As you might remember, in the binomial distribution, $\boldsymbol{n}$ is finite. Its smallest value of 0, and its largest value is, well, $\boldsymbol{n}$. The Poisson has no such limit (you may have noticed there was no $\boldsymbol{n}$ in the formula for the Poisson). That's one condition: $\boldsymbol{n}$ is infinite.

Now, recall that the average of the binomial distribution is

$$\mu = np$$

If this is the case, and $\boldsymbol{n}$ is growing without bound (becoming Poisson), then it must be true that $\boldsymbol{p}$ is getting very, very close to zero. That's the other condition: $\boldsymbol{p}$ is very small.

Let's do a little algebra, using the definition of combinations, the above binomial mean, and lots and lots of exponential rules:

$$\begin{aligned}
\binom{n}{x} p^x q^{n-x} &= \frac{n!}{(n-x)!x!}\left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x} \\
&= \frac{n!}{(n-x)!x!}\frac{\mu^x}{n^x}\left(1-\frac{\mu}{n}\right)^{n-x} \\
&= \frac{\mu^x}{x!}\frac{n!}{(n-x)!}\frac{1}{n^x}\left(1-\frac{\mu}{n}\right)^{n-x} \\
&= \frac{\mu^x}{x!}\frac{n!}{(n-x)!}\frac{1}{n^x}\left(1-\frac{\mu}{n}\right)^{n}\left(1-\frac{\mu}{n}\right)^{-x}
\end{aligned}$$

Now, remember the Poisson is boundless. Theoretically, we can't have those $\boldsymbol{n}$'s in there, can we? I mean, with the $\boldsymbol{n}$'s in there, it seems as though there's a top end, a limiting value. To alleviate this problem, we need calculus (in particular, limits), because we need to allow $\boldsymbol{n}$ to grow towards infinity. I know you probably haven't taken calculus, but I'll walk you through the computation, and I bet you'll be able to get it! This is how it looks (symbolically):

$$\lim_{n\to\infty}\frac{\mu^x}{x!}\frac{n!}{(n-x)!}\frac{1}{n^x}\left(1-\frac{\mu}{n}\right)^{n}\left(1-\frac{\mu}{n}\right)^{-x}$$

Now, we can rewrite the expression as such:

$$\lim_{n\to\infty}\frac{\mu^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}$$

Now, we can examine each of those four limit pieces:

1. $\lim_{n\to\infty}\frac{\mu^x}{x!}$, since it doesn't have any ***n***'s in it, stays $\frac{\mu^x}{x!}$.
2. $\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}$ is a bit tricky, but we can reason through it in this way: the first factor, $\frac{n!}{(n-x)!}$, is getting infinitely large as ***n*** gets infinitely large. The second factor $\frac{1}{n^x}$, however, is getting very, very small (since its denominator is getting infinitely large). Since the two factors are (roughly) reciprocal, their product is 1, so $\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}=1$
3. $\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{-\mu}{n}\right)^n=e^{-\mu}$, from the definition of Euler's number.
4. In $\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}$, since $\frac{\mu}{n}$ is getting very close to zero (why?), the number in parentheses is getting very close to 1 - 0, or 1, so $\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}=\lim_{n\to\infty}(1-0)^{-x}=\lim_{n\to\infty}1^{-x}=1$

We can now substitute these values back into the limit expression:

$$\lim_{n\to\infty}\frac{\mu^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}=\frac{\mu^x}{x!}\cdot 1\cdot e^{-\mu}\cdot 1=\frac{e^{-\mu}\mu^x}{x!}$$

This was exactly what we were trying to show!