<u>Justification for mean and standard deviation for *z*-scores</u>

As we have learned in class, for **z**-scores, $\mu_z = 0$ and $\sigma_z = 1$. This paper explains why.

In the justification below, I rely on a few basic formulas:

$z = \dfrac{x - \mu_{pop}}{\sigma_{pop}}$ (formula for **z**-score)	$\mu_{pop} = \dfrac{\sum x}{n}$ (formula for population mean)[1]
${\sigma_{pop}}^2 = \dfrac{\sum x^2}{n} - \dfrac{\left(\sum x\right)^2}{n^2}$ (formula for population variance)	$\sigma_{pop} = \sqrt{\dfrac{\sum x^2}{n} - \dfrac{\left(\sum x\right)^2}{n^2}}$ (formula for population standard deviation)

First, I will derive the mean:

$$\mu_z = \frac{\sum z}{n}$$

$$= \frac{\sum \frac{\left(x - \mu_{pop}\right)}{\sigma_{pop}}}{n}$$

$$= \frac{\frac{1}{\sigma_{pop}} \sum \left(x - \mu_{pop}\right)}{n}$$

$$= \frac{\frac{1}{\sigma_{pop}} \left(\sum x - \sum \mu_{pop}\right)}{n}$$

$$= \frac{\frac{1}{\sigma_{pop}} \left(\sum x - n\mu_{pop}\right)}{n}$$

$$= \frac{\frac{1}{\sigma_{pop}} \left(n\mu_{pop} - n\mu_{pop}\right)}{n}$$

$$= \frac{\frac{1}{\sigma_{pop}} (0)}{n} = 0$$

Which justifies why $\mu_z = 0$. Now for σ_z...

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[1] I will be using population formulas for this argument; sample formulas would yield similar results.

$$\sigma_z^{\ 2} = \frac{\sum z^2}{n} - \frac{\left(\sum z\right)^2}{n^2} = \frac{\sum z^2}{n} \ - \ 0$$

$$n\sigma_z^{\ 2} = \sum z^2 = \sum \left(\frac{x-\mu_{pop}}{\sigma_{pop}}\right)^2 = \frac{1}{\sigma_{pop}^{\ 2}} \sum \left(x^2 - 2x\mu_{pop} + \mu_{pop}^{\ 2}\right)$$

$$n\sigma_{pop}^{\ 2}\sigma_z^{\ 2} = \sum x^2 - \sum 2x\mu_{pop} + \sum \mu_{pop}^{\ 2}$$

$$n\sigma_{pop}^{\ 2}\sigma_z^{\ 2} = \sum x^2 - 2\mu_{pop}\sum x + n\mu_{pop}^{\ 2}$$

$$n\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n\mu_{pop}^{\ 2} = \sum x^2 - 2n\mu_{pop}^{\ 2}$$

$$n\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n\mu_{pop}^{\ 2} + 2n\mu_{pop}^{\ 2} = \sum x^2$$

$$n\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n\mu_{pop}^{\ 2} + 2n\mu_{pop}^{\ 2} = n\sigma_{pop}^{\ 2} + \frac{\left(\sum x\right)^2}{n}$$

$$n^2\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n^2\mu_{pop}^{\ 2} + 2n^2\mu_{pop}^{\ 2} = n^2\sigma_{pop}^{\ 2} + \left(\sum x\right)^2$$

$$n^2\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n^2\mu_{pop}^{\ 2} + 2n^2\mu_{pop}^{\ 2} = n^2\sigma_{pop}^{\ 2} + \left(n\mu_{pop}\right)^2$$

$$n^2\sigma_{pop}^{\ 2}\sigma_z^{\ 2} - n^2\sigma_{pop}^{\ 2} = n^2\mu_{pop}^{\ 2} - 2n^2\mu_{pop}^{\ 2} + n^2\mu_{pop}^{\ 2}$$

$$n^2\sigma_{pop}^{\ 2}\left(\sigma_z^{\ 2} - 1\right) = 0$$

$$\sigma_z^{\ 2} - 1 = 0$$ [□]

$$\sigma_z^{\ 2} = 1$$

$$\sigma_z = 1$$

Which justifies why $\sigma_z = 1$. **Q.E.D.**

[□] This step operates on the assumption that $\sigma_{pop} \neq 0$, which is fair.