## Quadratic Modeling Exercises

<table>
<thead>
<tr>
<th>Pages</th>
<th>Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>330 – 333</td>
<td>1,5-12,14,15,17,19,21</td>
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<tr>
<td></td>
<td>(for 19 and 21, you’re only deciding between linear and quadratic; we’ll get to exponential soon!)</td>
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</tbody>
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In class, we analyzed the function \( f(x) = x^2 \) to see why the second differences were equal. We then made the somewhat brash claim that, “Well, since the second differences are constant for this function, they’ll be thus for all quadratics.” (OK, I didn’t actually say that...but we pretty much bought it, didn’t we?)

Let’s be more mathematical...let’s use the general quadratic function \( f(x) = ax^2 + bx + c \), and show that the second differences are the same.

**E1.** What does \( f(x+1) - f(x) \) equal?

**E2.** What does \( f(x+2) - f(x+1) \) equal?

**E3.** What does \( f(x+3) - f(x+2) \) equal?

Each of those represented the first differences in the quadratic function...now, let’s look at the second differences.

**E4.** Subtract your result in E5 from your result in E6.

**E5.** Subtract your result in E6 from your result in E7.

**E6.** What do you notice about your answers in E8 and E9?

That oughta do it, although it still isn’t a formal proof. If you need (or want) more, and would like a few extra credit points, show me that the second difference is the same for the difference between any two sequential terms in a quadratic pattern. This sounds much harder than it actually is, and I’d be glad to help you get started! Swing by the office if you’re interested!

Now, refer back to exercise 5 in the text.

**E7.** Find the average rate of change (speed) of the football’s height between 0 and 0.5 seconds. Be sure to include the unit!

**E8.** Find the average rate of change of the football’s height between 0.5 and 1 seconds.

**E9.** Find the average rate of change of the football’s height between 1 and 1.5 seconds.

**E10.** Find the average rate of change of the football’s height between 1 and 2 seconds.

**E11.** Find the average rate of change of the football’s height between 1.5 and 2 seconds.

**E12.** Find the average rate of change of the football’s height between 1.75 and 2 seconds.

**E13.** Find the instantaneous rate of change of the football’s height at 2 seconds.
“E” Answers.

\[ E1. \quad f(x+1) = a(x+1)^2 + b(x+1) + c \quad \text{and} \quad f(x) = ax^2 + bx + c, \quad \text{so} \]

\[
\begin{align*}
  f(x+1) - f(x) &= a(x+1)^2 + b(x+1) + c - (ax^2 + bx + c) \\
                  &= ax^2 + 2ax + a + bx + b + c - ax^2 - bx - c \\
                  &= 2ax + a + b, \quad \text{or, if you like,} \quad a(2x+1) + b
\end{align*}
\]

I think you can handle E2 and E3!

\[ E4. \quad \text{I get } 2a. \quad \text{You?} \]

\[ E5. \quad \text{Hmmmmmm?} \]

\[ E6. \quad \text{Interesting! Which shows that the second differences are equal. Although it doesn’t prove it for all sequential values of } (x + h), \text{ it’s hopefully enough to convince you that there is a pattern (and inspire you to try the extra credit, if you so choose).} \]

\[ E7. \quad (\text{I’m calling the function } f(x) \text{ here, where } f \text{ is the height at } x \text{ seconds}) \]

\[
\begin{align*}
  f(0.5) - f(0) &= 22 - 6 \quad \text{feet} \\
  0.5 - 0 \quad \text{second} &= 32 \quad \text{feet} \\
  0.5 \quad \text{second} &= 32 \quad \text{second} \\
\end{align*}
\]

You can also use the best fit curve to do this!

\[ E9. \quad \text{0 feet per second (see why?)} \]

\[ E10. \quad -8 \text{ feet per second (why’s it negative?)} \]

\[ E13. \quad \text{OK, we need to do a little algebra here. From the average rate of change formula} \]

\[
\text{Average Rate Of Change} = \frac{f(x+h) - f(x)}{h}
\]

We need to get \( h \) to equal 0. If you remember, from class, we can’t simply plug in \( h = 0 \) (yet), since we get \( \frac{0}{0} \) (which is, essentially, living in sin). So, let’s finagle, algebraically:

\[
\begin{align*}
  \text{Average Rate Of Change (for our football problem)} \\
  &= \frac{f(x+h) - f(x)}{h} \\
  &= \frac{-16(x+h)^2 + 40(x+h)+6 - (-16x^2+40x+6)}{h} \\
  &= \frac{-16x^2 - 32xh - h^2 + 40x + 40h + 6 + 16x^2 - 40x - 6}{h} \\
  &= \frac{-32xh - h^2 + 40h}{h}
\end{align*}
\]
Ha! Notice that everything without an “h” term canceled. Onward!

\[
\frac{-32xh - h^2 + 40h}{h}
\]

\[
\Rightarrow \frac{h(-32x - h + 40)}{h}
\]

\[
= -32x - h + 40
\]

At this point, we can let h = 0, so we have

**Instantaneous** (\(h = 0\)) Rate Of Change (for our football problem) = \(-32x + 40\)

So, if \(x = 2\) (for 2 seconds into the experiment), we have an instantaneous speed of \(-32(2) + 40\), or **-24 feet per second**.
Quadratic Modeling Quizzes

Quiz 1.

(10 points) (w) Complete questions E8, E11, and E12 above. Make sure to show me the division you did (you can use E7 as a guide).
On earth, things fall under the force of gravity. This is fun! We can do things like have a catch, go sky diving, or rappel – all because of this wonderful force!

1. (2 points) For starters, Google “Why is there gravity” and tell me, in a couple sentences, what you find.

Now – even though we’re not entirely sure why there is gravity, we do know how it works. For example, suppose you need to know how high a cliff is (remember above when I say it’d be fun to rappel? Well, if you know how long your rope is, you’ll need to also know how tall a cliff is to make sure that it’s long enough to get down safely!). Check this out:

$$h = -16t^2 + v_0t + h_0$$

That’s a little equation that governs the position of any object thrown on earth. $h$ is your height after $t$ seconds, $v_0$ is your initial velocity, and $h_0$ is your initial height. The “-16” is a cool result that’s tied to the fact that gravity accelerates objects at a predictable rate.

Now – suppose you’re standing at the top of the cliff you want to rappel down. That cliff has a height...and you don’t know it. Let’s let that be the $h_0$ that we’re trying to find. To find it, we’re going to drop a rock over the edge of a cliff, and time how long it take for us to hear it hit the ground.

If we do that, then the height after $t$ seconds have passed will be 0 (since the rock is now on the ground). The initial velocity ($v_0$) will also be zero (since we’re dropping the rock and not throwing it down.

This means that the equation above can be rewritten as

$$0 = -16t^2 + h_0$$

2. (2 points) Explain why we can rewrite the equation in that way!

Use this equation to figure out the height of the cliff you’re standing on if the rock falls for

3. (2 points) 2 seconds.

4. (2 points) 3 seconds.

5. (2 points) 4 seconds.

---

1 We’ll assume that the cliff is fairly short, by “speed of sound” standards. When you’re climbing, you rarely carry a rope more than 250 feet long, anyway – which means that you wouldn’t be rappelling more than about 125 feet. Over a distance that short, we can ignore the effect of the speed of sound delay.
In class, we spoke of lots of ways in which quadratic relationships appear in the world around us. I’d like to explore another in this quiz – the **Golden Ratio**.

1. **(2 points)** Google “The Golden Ratio” and tell me what you find. Make sure to include its numerical value, in both exact (with the square root) and approximate form!

You may have learned (in your Googling) that The Golden Ratio gets a symbol: $\phi$. Why? Well, like $\pi$ and its lesser – known cousin, $e$, $\phi$ tends to show up quite a bit in the world (hence, it gets a cool name). Here are a couple of cool places $\phi$ shows up:

In the Mona Lisa, the ratio of the subject’s face (is her name Lisa?), length to width, is approximately $\phi$. In fact, there are many occurrences of $\phi$ in the Mona Lisa – this is just one of them.

In the Parthenon, the length to width ratio, again, is approximately $\phi$.

The nautilus shell’s long side to short side ratio is about $\phi$ as well.

(over, please!)
Here’s one of my favorites: In any regular (all sides and angles equal) pentagon, if you draw in the diagonals, you get a five pointed regular star. The lengths of some of those segments formed can be combined into a ratio of \( \phi \) for example, the ratio of DB to DC is \( \phi \).

2. (2 points) Find two other lengths in that figure that have, as their ratio, a value of \( \phi \). You might want to start by Googling “golden ratio pentagon” – unless you love doing geometry!

3. (4 points...2 for each) Find two other places the golden ratio occurs – one in nature (like the nautilus), and one in human – generated form (like the Mona Lisa or the Parthenon). For the human generated one, you may not use the Fibonacci Series. 😊

4. (2 points) In the Fibonacci Series (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc.), each successive element is formed by adding the previous two together. What does \( \phi \) have to do with this series? Again, Google is your friend!

“Why \( \phi = \frac{\sqrt{5} + 1}{2} \), you might ask. Good for you! It has to do with something in geometry call similarity...in a golden rectangle (that is, a rectangle whose side are in the golden ratio), if you cut off a square, the smaller rectangle you leave behind is also golden (and, thereby, similar to the larger one).

(extra 3 points) The large rectangle shown at left is “golden” – that is, the ratio of its long side to its short side (\( a + b \) to \( a \)) is \( \phi \). The small rectangle is also golden, that is the ratio of \( a \) to \( b \) is also \( \phi \). Give me a proof that demonstrates from where the \( \frac{\sqrt{5} + 1}{2} \) arises in the figure shown at right.
Quiz 4.

The data at left show various years’ worth of data on men’s cancer deaths (per 100,000 men) in the United states (source: USA Today)

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Deaths per 100K men</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>205.3</td>
</tr>
<tr>
<td>1985</td>
<td>212.6</td>
</tr>
<tr>
<td>1989</td>
<td>217.6</td>
</tr>
<tr>
<td>1993</td>
<td>212.1</td>
</tr>
<tr>
<td>1997</td>
<td>201.9</td>
</tr>
</tbody>
</table>

For these data, use your Excel Sheet to create the best – fit parabola (let x be the year and y be the number of men, per hundred thousand, who die from cancer).

1. **(4 points...2 for the correct number, and 2 for the correct unit!) (w)**

   Use your model to find the average rate of change in cancer deaths from 1990 to 1995. Remember that you can use your TI to do this; just fully explain what you did. You’ll have to use “per” twice in your answer!

2. **(1 point)** According to this model, how many US men (per 100K) should have died from cancer in 2011?

   According to the American Cancer Society report “Cancer Statistics” (http://onlinelibrary.wiley.com/doi/10.3322/caac.20121/pdf), in 2011, there were a total number of 300,430 male deaths from cancer in the US.

3. **(1 point)** How many US men are there? You’ll have to do a little online research to find this number (I Googled “population of US 2011”; the second result was a Census Bureau “quickfact” table that was perfect. You can assume the US is half men and half women. Also, assume all males are “men”; the ACS document appears to make this assumption)

4. **(1 point)** Time for some unit analysis...move the decimal point in your answer to part “c” 5 places to the left. That makes it now the number of US men in hundreds of thousands. Now, multiply this number by your answer in “b”. That should be the total number of US male deaths from cancer in 2011.

5. **(1 point)** What percent error is shown in your estimate? Use the formula

   \[
   \text{Percent error} = \frac{\text{regression predicted result in 2011} - \text{ACS estimated value in 2011}}{\text{ACS estimated value in 2011}}
   \]

6. **(1 point)** What could explain this fairly large error?
Years ago, someone told me “If you drop a penny off the top of the Empire State Building in New York, and it hits someone at ground level, they’ll die.”

#mindblown

Of course, now that I’m older, I don’t believe statements like that at face value anymore. So, we’ll model it! Go ahead and open up the spreadsheet that accompanies this quiz. When you do, you’ll see this!

![Image of Empire State Building](image_url)

Quiz 5.

1. (1 point) How long does it take the penny to hit the ground? You can round to the nearest hundredth of a second if you like. (hint: look at the table of values)

2. (1 point each) What’s the penny’s IROC
   a. 8 seconds before impact?
   b. 5 seconds before impact?
   c. 2 seconds before impact?
   d. 1 second before impact?
   e. 0.1 seconds before impact?
   f. 0.01 seconds before impact? At this point the “penny” disappears from the graph. It’s, essentially, “on the ground”.

Hopefully, this looks a tad familiar! It’s based off of the sheet we used for the in–class quiz about my bike speed. IN class, though, we worked through AROC (Average Rate Of Change) until we arrived at my IROC (Instantaneous Rate Of Change) at the moment I finished my 200 foot trip. This spreadsheet’s a tad different – it looks at the IROC of the penny after a certain number of seconds that it’s fallen. In other words, it’s like the penny has a speedometer and you get to read it.

So, just like we did in class, we’ll check the IROC of that penny after various points of time. And, just like in class, we’ll measure those points of time through the reference of “seconds until impact.”
So now you know how fast it’s moving when it hits the ground (or, someone standing near the ground). And that’s pretty darned fast! However, speed’s only part of the process of understanding if someone would die if they were struck. Weight, although it has nothing to do with how fast the penny falls (everything on earth falls at the same acceleration), has everything to do with how the penny feels when it hits (imagine dropping an anvil and a crumpled piece of paper at the same time. Both would hit the ground at the same time, but...well, you get the idea).

So, let’s do some physics! Since the penny (used this way) is a projectile, it’s basically a “bullet”. Start by visiting this page! http://www.reloadammo.com/footpound2.htm. We’ll need that in a minute!

3. **(1 point)** Now, find the weight of a penny. Then convert that weight to grains. Apparently, grains” is the preferred unit for ballistics.

4. **(1 point)** Convert your speed in MPH to feet per second. That’s the unit the calculator needs.

5. **(1 point)** How many “foot pounds of energy” does the bullet have as it hits the ground?

That number might seem a little cryptic to you (heck, it sure did to me). I did some Googling, and found this: http://www.chuckhawks.com/rifle_ballistics_table.htm. Our little penny doesn’t even come close to getting the amount of energy (even after 1000 feet) that any of those rifles have. Then, I went back to this: https://en.wikipedia.org/wiki/Muzzle_energy. There, I discovered that our little penny, basically, is like an undersized pellet fired from an AirSoft gun. It might sting, but it sure as heck isn’t gonna kill you.

So there you go! Myth **busted**!

6. **(extra 2 points)** I made an assumption about the acceleration of our penny that...well, it just isn’t true. What was that assumption (and why does it make the “dropped pennies are lethal” argument even less meaningful)? Hint: we mentioned it in class, but it didn’t apply to the bike scenario.